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SYMBOLIC DYNAMICS FROM SIGNED MATRICES

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Abstract. We consider a method for assigning a sofic shift to a (not necessarily nonnegative integer) matrix by associating to it a directed graph with some vertices labelled 1 and the rest 2 (the decomposition of the vertices is arbitrary - in applications the choice should be natural). We can detect positive topological entropy for this sofic shift by comparing the characteristic polynomial of the original matrix to those for the matrices for the restrictions of the shifts to each piece (1 and 2). Our main application is to the use of the Conley index to detect symbolic dynamics in isolated invariant sets, and is an extension of a result by Carbinatto, Kwapisz, and Mischaikow.

1. **Introduction.** If M is a square matrix with nonnegative integer entries, then we can associate to it a subshift of finite type by considering it as the adjacency matrix for a directed graph. In this case, the topological entropy of the subshift will be equal to the log of the spectral radius of M. In this paper, we consider subshifts associated to matrices with arbitrary entries (the associated shift will be that corresponding to the matrix with each nonzero entry of the original matrix replaced by 1). Here the spectral radius gives no information about the entropy, because of the change in absolute value of the entries and cancellation of signed terms.

We can factor this shift onto a sofic shift by labelling some of the vertices of the associated graph 1 and the others 2. We can detect positive entropy for this sofic shift by comparing the characteristic polynomial of the original matrix to those for the matrices for the restrictions of the shifts to each piece (1 and 2). If our matrix represents a homology map induced by a self-map of a topological space, then in some cases the vertex decomposition will be natural, and the resulting sofic shift will have dynamical significance.

Our main application is to the detection of symbolic dynamics in isolated invariant sets. Let N be an isolating neighborhood for a map f. If we can decompose N into the disjoint union of compact sets N_1 and N_2, then we can relate the dynamics on the maximal invariant set $\operatorname{Inv} N$ to the shift on two symbols by noting which component of N each iterate of a point $x \in \operatorname{Inv} N$ lies in. In [2], it is shown that if the eigenvalues of the Conley index map for N differ in a certain way from those for N_1 and N_2, then there exists a positive integer d such that f^d factors onto the full shift on two symbols. The number d is not specified, however, so this result provides no estimate for the topological entropy of f. With some additional hypotheses, we provide an upper bound for d, and thus a lower bound for $h_{top}(f)$.

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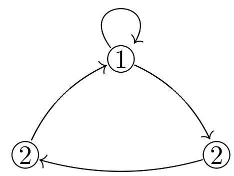


FIGURE 1. Graph corresponding to a sofic shift

2. **Sofic shifts.** Sofic shifts are a generalization of subshifts of finite type originally developed by Weiss in [11]. Our discussion is taken from that in [8], which contains proofs and more formal definitions.

We can think of a subshift of finite type as corresponding to a directed graph, with vertex set $\{v_i\}$. We say that $v_i \to v_j$ (v_j can follow v_i) if there is an edge running from vertex v_i to vertex v_j.

The (one-sided) subshift of finite type associated to a graph consists of all infinite sequences of vertices $(v_{i_0}, v_{i_1}, \dots)$ such that $v_{i_j} \to v_{i_{j+1}}$ for all $j \geq 0$, along with the shift map σ defined by $\sigma((v_{i_0}, v_{i_1}, \dots)) = (v_{i_1}, v_{i_2}, \dots)$. Note that many different graphs may correspond to the same subshift of finite type.

Given a graph, we define its adjacency matrix $A = (a_{ij})$ by setting

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \to v_i, \\ 0 & \text{otherwise.} \end{cases}$$

We construct a sofic shift in the same way, except that we now give each vertex v_i a label, $L(v_i)$ (some of the vertices may have the same label). The sofic shift associated to a labelled directed graph consists of all infinite sequences of labels $(L(v_{i_0}), L(v_{i_1}), \dots)$ such that $v_{i_j} \to v_{i_{j+1}}$ for all $j \geq 0$, along with the shift map σ. See Figure 1. (That graph corresponds to the even shift, i.e., the space of all sequences of 1's and 2's with each block of 2's having even length.)

We will consider primarily sofic shifts that are subshifts of (Σ_2^+, σ), the full (one-sided) shift on the two symbols $\{1, 2\}$.

3. **Sofic shifts from signed matrices.** In making the following definitions, we will be considering a real $n \times n$ matrix M as representing a directed graph with weighted edges. The vertices of the graph are the basis elements $v_1, \dots, v_n$. There is an edge running from v_i to v_j if $M_{ji} \neq 0$, and its weight is M_{ji}. Thus M-paths for the matrix correspond to paths in the graph, and the weight of a path is the product of the weights of its component edges. The shift associated to M will be that for the adjacency matrix constructed by replacing each nonzero entry of M by 1. Later we will consider the sofic shift that we get by sorting the vertices into two distinct classes.

More formally, let V be an n-dimensional vector space with basis $\{v_1, \dots, v_n\}$, and M an $n \times n$ matrix, which we can consider as representing a map on V with the given basis. For $1 \leq j \leq n$, define projection maps $p_j : V \to \mathbb{R}$ as follows: if $v = \sum_{i=1}^n a_i v_i$, set $p_j(v) = a_j$.

Definition 3.1. An *M-path* $\gamma = (\gamma_0, \dots, \gamma_m) = (v_{r_0}, \dots, v_{r_m})$ *from* v_i *to* v_j *of length* m (where m is a positive integer) is an ordered $(m+1)$-tuple of elements of $\{v_1, \dots, v_n\}$ with $\gamma_0 = v_i$ and $\gamma_m = v_j$ such that $p_{r_{k+1}}(M(\gamma_k)) \neq 0$ for $0 \leq k < m$. (Note that a path must have length at least one, i.e., a single element is not a path.)

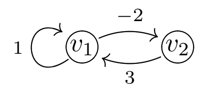


FIGURE 2. Weighted graph corresponding to M

We denote by $l(\gamma)$ its length m and define its *weight* $w(\gamma)$ to be $\prod_{k=0}^{m-1} p_{r_{k+1}}(M(\gamma_k))$. We will also consider infinite M-paths $\gamma = (\gamma_0 = v_i, \gamma_1, \gamma_2, \dots)$.

An *M-loop λ at v_i* is an M-path from v_i to v_i.

Example 3.2. Let $M = \left(\begin{smallmatrix} 1 & 3 \\ -2 & 0 \end{smallmatrix}\right)$, corresponding to the graph in Figure 2 (the number along each edge represents its weight). Then (v_1, v_2, v_1) is an M-loop at v_1 of length two and weight -6, since $p_2(M(v_1)) = p_2(v_1 - 2v_2) = -2$ and $p_1(M(v_2)) = p_1(3v_1) = 3$.

Observe that if $p_j(M^m(v_i)) \neq 0$, then there must be an M-path of length m from v_i to v_j. The converse is not true, as the following example shows. (It is true if M is nonnegative.)

Example 3.3. Let $M = \left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. Then for any $m \geq 1$ and any i and j, there is an M-path of length m from v_i to v_j. But $M^m = 0$ for $m \geq 2$.

Next, we see that there is a strong relation between M-loops of a matrix M and its characteristic polynomial.

Proposition 3.4. *For $1 \leq i, j \leq n$ and $m \geq 1$, $(M^m)_{ji} = \sum w(\gamma)$, where the sum is taken over all M-paths from v_i to v_j of length m.*

Proof. First, observe that

$$M(v_i) = \sum_{k=1}^{n} M_{ki} v_k = \sum_{\{k \mid M_{ki} \neq 0\}} M_{ki} v_k = \sum w(\gamma_k) v_k,$$

where the last sum is taken over all k such that there is a (necessarily unique) path γ_k from v_i to v_k of length one. The result then follows from linearity and the fact that every path is a concatenation of paths of length one. (Compare [9, Lemma III.2.2], which is the same proposition for nonnegative adjacency matrices.) □

Corollary 3.5. *For $1 \leq i \leq n$ and $m \geq 1$, $(M^m)_{ii} = \sum w(\lambda)$, where the sum is taken over all M-loops at v_i of length m.*

Corollary 3.6. *The characteristic polynomial of M depends only on the set of M-loops λ with $1 \leq l(\lambda) \leq n$ and their weights $w(\lambda)$.*

Proof. Newton's formula ([4, §92]) tells us that knowledge of $\operatorname{tr}(M^m)$ for $1 \leq m \leq n$ is equivalent to knowledge of the characteristic polynomial of M. Since $\operatorname{tr}(M^m) = \sum_{i=1}^{n} (M^m)_{ii}$, the result follows from Corollary 3.5. □

Definition 3.7. Let v_i and v_j be basis elements of V. We say that v_i and v_j are in the same *M-communicating class* (or write $v_i \sim_M v_j$) if there exist an M-loop at v_i containing v_j and an M-loop at v_j containing v_i, or if $i = j$. (This is a generalization of the definition of communicating classes given in [8, §4.4].)

Proposition 3.8. *The relation $\sim_M$ is an equivalence relation on the set $\{v_1, \dots, v_n\}$.*

(a) M (b) M_{red}

FIGURE 3. Weighted graphs corresponding to M and M_{red}

Proof. It is clear that $\sim_M$ is reflexive and symmetric. We must show that it is transitive. Assume that $v_i \sim_M v_j$ and $v_j \sim_M v_k$. If any two of the vertices are equal, then it is immediate that $v_i \sim_M v_k$. So assume that all three are different. Let $\lambda = (\lambda_0 = v_i, \lambda_1, \dots, \lambda_p = v_j, \dots, \lambda_q = v_i)$ be an M-loop at v_i containing v_j, and $\mu = (\mu_0 = v_j, \dots, \mu_r = v_k, \dots, \mu_s = v_j)$ an M-loop at v_j containing v_k. Then $(\lambda_0, \lambda_1, \dots, \lambda_p = \mu_0, \mu_1, \dots, \mu_r = v_k, \dots, \mu_s = \lambda_p, \lambda_{p+1}, \dots, \lambda_q)$ is an M-loop at v_i containing v_k, so $v_i \sim_M v_k$. □

Thus $\{v_1, \dots, v_n\}$ is partitioned by $\sim_M$ into the disjoint union of M-communicating classes $\coprod W_i$. We will at times abuse notation by considering each W_i as a vector subspace of V (namely the subspace spanned by the vectors contained in W_i).

Definition 3.9. An M-communicating class W_i is *M-nontrivial* if for every v_j in W_i, there is an M-loop at v_j (i.e., W_i does not consist of a single vertex with no loop through it). W_i is *M-trivial* otherwise (i.e., W_i is a single vertex with no loop through it).

Example 3.10. Let $M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}$, with corresponding graph in Figure 3(a). The M-trivial M-communicating classes are $\{v_2\}$ and $\{v_5\}$ and the M-nontrivial classes are $\{v_1\}$ and $\{v_3, v_4\}$.

Given M, we define M_{red}, the *reduced matrix*, by setting

$$(M_{\text{red}})_{ij} = \begin{cases} M_{ij} & \text{if } v_i \sim_M v_j, \\ 0 & \text{if } v_i \not\sim_M v_j. \end{cases}$$

That is, the graph represented by M_{red} is the graph represented by M with all edges between vertices in different M-communicating classes removed.

Example 3.11. Let M be the matrix defined in Example 3.10. Then $M_{\text{red}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Its corresponding graph is in Figure 3(b).

For each M-communicating class we define the square matrix M_{W_i} to be the submatrix of M corresponding to the induced map from W_i to W_i. The graph represented by M_{W_i} corresponds to the subgraph of M's graph consisting of the vertices of W_i and the edges among them.

Remark 3.12. $M_{\text{red}} = \oplus M_{W_i}$ (possibly after permuting the order of the basis elements).

Example 3.13. Let M and M_{red} be as in Examples 3.10 and 3.11. Then $M_{\text{red}} = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & 3 \\ 1 & 0 \end{matrix}} & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & \boxed{0} \end{pmatrix}$, where each boxed submatrix is an M_{W_i}.

Given a square matrix A, let $C_A(X)$ denote its characteristic polynomial.

Lemma 3.14. *$C_M(X) = C_{M_{\text{red}}}(X) = \prod_{W_i} C_{M_{W_i}}(X)$, where the product is taken over all M-communicating classes W_i.*

Proof. To prove the first equality, note that we removed only edges joining vertices in different M-communicating classes to produce M_{red} from M. Thus M and M_{red} have exactly the same loops with the same weights, and the equality follows from Corollary 3.6. The second equality follows from Remark 3.12. □

Proposition 3.15. *If an M-communicating class W_i is M-trivial, then $M_{W_i} = (0)$, the 1×1 matrix whose entry is 0.*

Proof. By definition, $W_i = \{v_j\}$, where v_j is a vertex with no M-loop through it. Thus $M_{W_i} = (M_{jj}) = (0)$. □

We now assume that we decompose V into two parts. We can then associate a sofic shift to a matrix by labelling each vertex according to which part it is in and looking at the resulting itineraries of paths. More formally, let V_1 and V_2 be positive-dimensional vector spaces, with bases $\{v_1^1, \ldots, v_{n_1}^1\}$ and $\{v_1^2, \ldots, v_{n_2}^2\}$ respectively. Let M_1 be an $n_1 \times n_1$ matrix representing a map on V_1, and M_2 an $n_2 \times n_2$ matrix representing a map on V_2. Define the $(n_1+n_2) \times (n_1+n_2)$ matrix $M := M_1 \oplus M_2$, so that M has the form $\left(\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right)$.

Let $M' : V_1 \oplus V_2 \to V_1 \oplus V_2$ be another $(n_1+n_2) \times (n_1+n_2)$ matrix such that $M'|_{V_1} : V_1 \to V_1 = M_1$ and $M'|_{V_2} : V_2 \to V_2 = M_2$ (so that M' has the form $\left(\begin{array}{c|c} M_1 & P \\ \hline Q & M_2 \end{array}\right)$).

Remark 3.16. Any M-path is also an M'-path, since the graph represented by M' consists of the graph represented by M plus additional edges, represented by the matrices P and Q, linking vertices in V_1 to vertices in V_2 and vice versa.

Define a map $\pi : \{v_1^1, \ldots, v_{n_1}^1, v_1^2, \ldots, v_{n_2}^2\} \to \{1, 2\}$ by setting $\pi(v_i^j) = j$ for $j = 1, 2$ and $i = 1, \ldots, n_j$. If $\gamma = (\gamma_0, \ldots, \gamma_l)$ is an M'-path of length l (where l is either a positive integer or infinity), define $\psi(\gamma)$, the *itinerary* of γ, to be the element of $\{1, 2\}^l$ given by

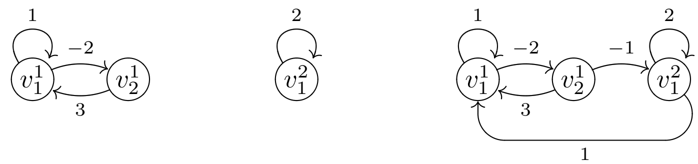
$$\psi(\gamma) = (\pi(\gamma_0), \pi(\gamma_1), \ldots, \pi(\gamma_l)).$$

If γ is an M'-loop (i.e., l is finite and $\gamma_0 = \gamma_l$), then define its *nonredundant itinerary* $\bar{\psi}(\gamma)$ by setting

$$\bar{\psi}(\gamma) = (\pi(\gamma_0), \pi(\gamma_1), \ldots, \pi(\gamma_{l-1})).$$

(The redundancy refers to the equality of the first and last entries of the regular itinerary $\psi(\gamma)$.)

Example 3.17. Let $M_1 = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}$ (with corresponding graph in Figure 4(a)), and $M_2 = (2)$ (see Figure 4(b)), so that $M = \left(\begin{array}{cc|c} 1 & 3 & 0 \\ -2 & 0 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$ (the graph



(a) M_1 (b) M_2 (c) M'

FIGURE 4. Weighted graphs corresponding to M_1, M_2, and M'

corresponding to M is just the disjoint union of the graphs for M_1 and M_2). Let $M' = \left(\begin{array}{cc|c} 1 & 3 & 1 \\ -2 & 0 & 0 \\ \hline 0 & -1 & 2 \end{array}\right)$, with corresponding graph in Figure 4(c). The M'-loop $(v_1^1, v_2^1, v_1^2, v_1^1)$ has itinerary $(1, 1, 2, 1)$ and nonredundant itinerary $(1, 1, 2)$.

Let $S(M')$ be the set of all infinite M'-paths. Define $(\Gamma_{M'}, \sigma_{M'})$ to be the sofic shift given by the restriction of (Σ_2^+, σ) to $\psi(S(M'))$.

We can make similar definitions for (Γ_M, σ_M), but it is contained entirely in the set $\{(1, 1, 1, \dots)\} \cup \{(2, 2, 2, \dots)\}$ and is not very interesting. By Remark 3.16, $(\Gamma_{M'}, \sigma_{M'})$ contains (Γ_M, σ_M), and possibly much more as well. We wish to obtain conditions on the characteristic polynomials of M and M' guaranteeing that $(\Gamma_{M'}, \sigma_{M'})$ will have positive topological entropy. The following lemma will be a key to our analysis.

Lemma 3.18. *Let x be a vertex of M', i.e., an element of $\{v_1^1, \dots, v_n^1, v_1^2, \dots, v_m^2\}$, and let l be a positive integer. Let $\lambda = (\lambda_0, \dots, \lambda_l)$ and $\mu = (\mu_0, \dots, \mu_l)$ be M'-loops at x of length l. If $\bar{\psi}(\lambda) \neq \bar{\psi}(\mu)$, then the lth power of $(\Gamma_{M'}, \sigma_{M'})$ factors onto the full 2-shift. Thus $h_{top}(\sigma_{M'}) \geq \frac{\log 2}{l}$.*

Proof. Define a map ϕ from $\Gamma_{M'}$ to Σ_2^+ as follows. If $\tau = (\tau_0, \tau_1, \dots)$ ($\tau_i \in \{1, 2\}$ for all i) is an element of $\Gamma_{M'}$, let $\phi(\tau) = (\tau_0, \tau_l, \tau_{2l}, \dots)$. Clearly $\phi \circ \sigma_{M'}^l = \sigma \circ \phi$. Since $\Gamma_{M'}$ contains the subshift generated by the concatenations of $\bar{\psi}(\lambda)$ and $\bar{\psi}(\mu)$, and $\bar{\psi}(\lambda) \neq \bar{\psi}(\mu)$, ϕ is surjective. Therefore, since $h_{top}(\sigma) = \log 2$, $h_{top}(\sigma_{M'}) \geq \frac{\log 2}{l}$. □

Thus we want to determine whether there exist M'-loops of the same length at the same vertex with different nonredundant itineraries.

Definition 3.19. Let $P(X) \in \mathbb{Z}[X]$ be a polynomial of degree p, and b the multiplicity of zero as a root of $P(X)$. Define the polynomial $\bar{P}(X)$ of degree $n - b$ to be $P(X)/X^b$. (For example, if $P(X) = X^3 + 2X$, then $\bar{P}(X) = X^2 + 2$, and if $Q(X)$ is a power of X, then $\bar{Q}(X) = 1$.)

Proposition 3.20. $\bar{C}_M(X) = \prod_{W_i} \bar{C}_{M_{W_i}}(X)$, *where W_i runs over all M-nontrivial M-communicating classes.*

Proof. This follows from Lemma 3.14 and Proposition 3.15. □

Remark 3.21. If $\gamma = (\gamma_0, \dots, \gamma_r)$ is an M'-path and $\lambda = (\lambda_0, \dots, \lambda_s)$ is an M'-loop with $\lambda_0 = \lambda_s = \gamma_i$ for some i, $0 \leq i \leq r$, then we can link λ with γ at γ_i to produce a new M'-path of length $r + s$ from γ_0 to γ_r (namely $(\gamma_0, \dots, \gamma_i = \lambda_0, \lambda_1, \dots, \lambda_s = \gamma_i, \gamma_{i+1}, \dots, \gamma_r)$). In particular, if γ and λ are both M'-loops at λ_0, we can link γ and λ at γ_0 to produce a new M'-loop at λ_0.

Remark 3.22. Let A be a $p \times p$ matrix. We call an A-loop *minimal* if it does not contain any proper subloops. A minimal A-loop λ must have length less than or equal to p (since there are only p different vertices, if the length of λ is greater than p there must be vertices $\lambda_i = \lambda_j$ with $i < j < l(\lambda)$, and $(\lambda_i, \dots, \lambda_j)$ is a proper subloop). Any A-loop is made up of linked minimal A-loops. In particular, any A-loop at a vertex v_i contains a (not necessarily proper) subloop at v_i of length less than or equal to p.

Let $d = \max(\dim V_1, \dim V_2) = \max(n_1, n_2)$.

Theorem 3.23. *If $\bar{C}_M(X)$ is not a factor of $\bar{C}_{M'}(X)$, then there exists a positive integer $k \le (n_1 + n_2)d$ such that $\sigma_{M'}^k$ factors onto the full two-shift. Thus $h_{top}(\sigma_{M'}) \ge \frac{\log 2}{k}$.*

Proof. We will show that there is a new M'-loop λ (i.e., λ is not an M-loop) at a vertex that also has an "old" M-loop μ through it, and, after a little more work, apply Lemma 3.18.

Assume that there is no such M'-loop λ, i.e., that no new M'-loop passes through a vertex that has an M-loop through it. In other words, then, all of the new M'-loops pass through only vertices that were in M-trivial M-communicating classes. Therefore each M-nontrivial M-communicating class W_i is also an M'-nontrivial M'-communicating class. Since there are no new M'-loops involving vertices in W_i, we have that $C_{M_{W_i}}(X) = C_{M'_{W_i}}(X)$, by Corollary 3.6. But then Proposition 3.20 implies that $\bar{C}_M(X)$ divides $\bar{C}_{M'}(X)$, contrary to hypothesis.

So let λ be a new M'-loop at x, and μ an M-loop (which by Remark 3.16 is also an M'-loop) at x. We may assume that $l(\lambda) \le n_1 + n_2$ (Remark 3.22). Also, since $M = M_1 \oplus M_2$, μ is actually either an M_1-loop or an M_2-loop. Thus we can also assume that $l(\mu) \le d$. Let $k = \mathrm{LCM}(l(\lambda), l(\mu)) \le l(\lambda)l(\mu) \le (n_1 + n_2)d$. Then by linking λ with itself $\frac{k}{l(\lambda)}$ times and μ with itself $\frac{k}{l(\mu)}$ times, we can produce new M'-loops at x, λ_0 and μ_0, with $l(\lambda_0) = l(\mu_0) = k$. (For example, if $\lambda = (v_1^1, v_1^2, v_1^1)$ and $\mu = (v_1^1, v_1^1)$, then $k = 2$, $\lambda_0 = \lambda$, and $\mu_0 = (v_1^1, v_1^1, v_1^1)$.)

Now, since λ is an M'-loop but not an M-loop, and all the edges gained passing from M to M' link vertices in V_1 to vertices in V_2 or vice versa, $\bar{\psi}(\lambda)$ contains both 1's and 2's, and the same is true of $\bar{\psi}(\lambda_0)$. Similarly, since μ and μ_0 are M-loops, and M has no edges between vertices in V_1 and vertices in V_2, $\bar{\psi}(\mu_0)$ contains either only 1's or only 2's. In particular, $\bar{\psi}(\lambda_0) \ne \bar{\psi}(\mu_0)$, and we can apply Lemma 3.18. □

Thus if we lose a nonzero eigenvalue when we pass from M to M', the shift $(\Gamma_{M'}, \sigma_{M'})$ must have positive entropy. However, this is not necessarily true if we *gain* a nonzero eigenvalue, as the following example shows.

Example 3.24. Let $M' = \left(\begin{array}{c|cc} 0 & 2 & -1 \\ \hline 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)$, with corresponding graph in Figure 5. Then $\bar{C}_M(X) = 1$, while $\bar{C}_{M'}(X) = X^2 - 1$. However, any M'-loop λ at v_1^1 must have length $2l$ for some l and $\bar{\psi}(\lambda) = (1,2)^l$. Similarly, any M'-loop μ at v_i^2 $(i = 1, 2)$ must have length $2l_0$ and $\bar{\psi}(\mu) = (2,1)^{l_0}$. Thus

$$(\Gamma_{M'}, \sigma_{M'}) \subset \{(1,2,1,2,\dots)\} \cup \{(2,1,2,1,\dots)\}$$

and therefore has zero topological entropy.

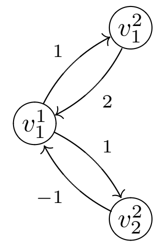


FIGURE 5. Graph corresponding to M'

The problem in this example is that every M'-loop through a given vertex has the same nonredundant itinerary. It turns out that the characteristic polynomial of such a matrix must have a certain form. So if $\bar{C}_M(X)$ divides $\bar{C}_{M'}(X)$ but the polynomial $\bar{C}_{M'}(X)/\bar{C}_M(X)$ does not have that form, we still get a shift of positive topological entropy. To make this precise, we need the following definition.

Definition 3.25. A polynomial $P(X) \in \mathbb{Z}[X]$ is *cyclic* if it is also a polynomial in a higher power of X, i.e., if there exist an integer $k > 1$ and a polynomial Q such that $P(X) = Q(X^k)$. (Thus $X^4 + X^2 + 1 = (X^2)^2 + (X^2) + 1$ is cyclic, while $X^2 + X$ is not.)

Definition 3.26. Define a function $B : \mathbb{N} \to \mathbb{R}$ by

$$B(k) = \frac{(k+1)^2}{4}\left(\frac{(2+k)(k+1)^2}{8}\right)^{\log_2 \frac{(k+1)^2}{4}}.$$

As before, let $d = \max(\dim V_1, \dim V_2) = \max(n_1, n_2)$. Let b be the multiplicity of zero as a root of $C_M(X)$.

Theorem 3.27. *If $\bar{C}_M(X)$ divides $\bar{C}_{M'}(X)$ but the polynomial $\bar{C}_{M'}(X)/\bar{C}_M(X)$ is not a product of cyclic polynomials, then there exists an integer $k \leq \max((n_1 + n_2)d, B(b))$ such that $\sigma_{M'}^k$ factors onto the full two-shift.*

Remark 3.28. The part of the estimate for k involving b is rather crude, as we shall see. For small b we can get better estimates by inspection. If $b = 2$, then we can replace the second term of the maximum by 2. If $b = 3$, we can replace it by 6. And if $b = 4$, we can replace it by 12.

Proof. Again, we will try to apply Lemma 3.18. If there is a new M'-loop at the same vertex as an old M-loop, then we may proceed as in the proof of Theorem 3.23, and we are done. So assume that all new M'-loops contain only vertices that were M-trivial. Then we will see that the hypothesis of the theorem implies that at some vertex there are two new M'-loops of relatively prime length. It is then easy to see that we can produce from these two M'-loops that satisfy the hypotheses of Lemma 3.18, completing the proof.

Let $\{W_j\}$ be the set of all M'-nontrivial M'-communicating classes containing vertices that were M-trivial (i.e, the vertices in $\{W_j\}$ are all of those contained in the new M'-loops). Then the hypothesis of the lemma implies that for some W_j, the polynomial $\bar{C}_{M'_{W_j}}(X)$ is noncyclic.

Lemma 3.29. *Let A be a $p \times p$ matrix (with $p \geq 2$) such that for any pair of vertices there is an A-loop joining them, and such that $\bar{C}_A(X)$ is noncyclic. Then at some vertex there are two A-loops of relatively prime length, with the length of* *one less than or equal to* $\frac{(p+1)^2}{4}$ *and the length of the other less than or equal to* $(\frac{(2+p)(p+1)^2}{8})^{\log_2 \frac{(p+1)^2}{4}}$.

We will postpone the proof of this lemma.

Since W_j consists of vertices that were M-trivial, by Lemma 3.14 and Proposition 3.15 W_j contains at most b vertices. Thus the preceding lemma tells us that there exist new M'-loops λ and μ at some vertex of W_j with $l(\lambda) \le \frac{(b+1)^2}{4}$, $l(\mu) \le (\frac{(2+b)(b+1)^2}{8})^{\log_2 \frac{(b+1)^2}{4}}$, and $\mathrm{GCD}(l(\lambda), l(\mu)) = 1$.

Let λ' be the M'-loop obtained by linking λ with itself $l(\mu)$ times, and μ' that obtained by linking μ with itself $l(\lambda)$ times. Observe that

$$l(\lambda') = l(\mu') \le \frac{(b+1)^2}{4} \left(\frac{(2+b)(b+1)^2}{8}\right)^{\log_2 \frac{(b+1)^2}{4}} = B(b).$$

We wish to apply Lemma 3.18, so we need to show that $\bar{\psi}(\lambda') \neq \bar{\psi}(\mu')$.

Assume without loss of generality that $\pi(\lambda_0) = 1$. Since μ is a new M'-loop, it must contain vertices from both V_1 and V_2. Thus there is a j, $0 \le j < l(\mu)$, such that $\pi(\mu_j) = 2$. Now, $\lambda'_{l(\lambda)t} = \lambda_0$ for $t = 0, \ldots, l(\mu)$, and $\mu'_{l(\mu)s+j} = \mu_j$ for $s = 0, \ldots, l(\lambda) - 1$. Since $l(\lambda)$ and $l(\mu)$ are relatively prime, the Chinese Remainder Theorem ([3, Th. 24]) tells us that there exists a q, $0 \le q < l(\lambda)l(\mu)$, such that $q \equiv 0 \pmod{l(\lambda)}$ and $q \equiv j \pmod{l(\mu)}$. In other words, $q = l(\lambda)t$ for some t, $0 \le t < l(\mu)$, and $q = l(\mu)s + j$ for some s, $0 \le s < l(\lambda)$. Thus $\pi(\lambda'_q) = \pi(\lambda_0) = 1$ and $\pi(\mu'_q) = \pi(\mu_j) = 2$. So $\bar{\psi}(\lambda') \neq \bar{\psi}(\mu')$, completing the proof. □

We still must prove Lemma 3.29.

Proof of Lemma 3.29. We will need the following lemmas.

Lemma 3.30. *Let A be as in the statement of Lemma 3.29. Then there is no integer $t > 1$ such that t divides the length of every A-loop.*

Proof. Assume that there is a $t > 1$ such that the length of every A-loop is divisible by t. Then $\mathrm{tr}(A^j) = 0$ for $j \not\equiv 0 \pmod t$, by Corollary 3.5. Thus Newton's formula ([4, §92]) tells us that $C_A(X)$ has the form $X^s P(X^t)$ for some $s \ge 0$ and some polynomial P with nonzero constant term. But then $\bar{C}_A(X) = P(X^t)$, contrary to hypothesis. □

So for any integer $t > 1$, there exists a minimal A-loop whose length is not divisible by t (otherwise, since every A-loop is made up of minimal A-loops linked together (Remark 3.22), every A-loop would have length divisible by t).

Lemma 3.31. *There is an A-loop λ of length less than or equal to $\frac{(p+1)^2}{4}$ containing all p vertices.*

Proof. For each $i \neq j$, let μ^{ij} be the shortest A-path from v_i to v_j. Let μ be a μ^{ij} of maximal length, which length we will denote by k. Let q be the number of vertices that μ misses. Then $k = p - q - 1$ (since μ is the shortest A-path between the vertices that it joins, it contains no A-loops, and hence hits each of the $p - q$ vertices that it contains exactly once). Join to μ the shortest A-path starting at the end of μ and ending at the first of the q vertices missed by μ. From the end of this new A-path join the shortest A-path to the second vertex missed by μ, then the third, and so on, until we have an A-path from the initial vertex of μ to the last of the vertices missed by μ. This A-path contains all p vertices. Finally, join the shortest A-path from the end back to the initial vertex of μ, and call the resulting A-loop λ.

We must estimate the length of λ. We started with μ, and added to it $q+1$ A-paths, each of length less than or equal to k. Thus

$$l(\lambda) \leq l(\mu) + (q+1)k = (q+2)k = (q+2)(p-q-1).$$

The function $f(x) = (x+2)(p-x-1)$ has its absolute maximum at $x = \frac{p-3}{2}$, and $f(\frac{p-3}{2}) = \frac{(p+1)^2}{4}$. Thus $l(\lambda) \leq \frac{(p+1)^2}{4}$. $\square$

Let λ be such an A-loop at some vertex x. Since λ contains every vertex, we can link any other A-loop with it. We will produce a new A-loop μ at x of length less than or equal to $\left(\frac{(2+p)(p+1)^2}{8}\right)^{\log_2 \frac{(p+1)^2}{4}}$ such that $\mathrm{GCD}(l(\lambda), l(\mu)) = 1$.

Let $l(\lambda) = p_1^{d_1} \dots p_k^{d_k}$ be the prime factorization of $l(\lambda)$. For each i, there is an A-loop ν^i of length q_i such that $p_i \not| q_i$ and $p_j | q_i$ for every $j \neq i$ (take any minimal A-loop whose length is not divisible by p_i and link it with itself $\prod_{j \neq i} p_j$ times).

We will define μ inductively. Let $\mu^0 = \lambda$. Given μ^{s-1}, construct μ^s by linking $(\prod_{1 \leq i < s}(p_i^{d_i} + q_i))(\prod_{k \geq j > s} p_j^{d_j})$ copies of ν^s to μ^{s-1}. Let $\mu = \mu^k$.

By construction, $l(\mu) = \prod_{i=1}^k (p_i^{d_i} + q_i)$, which is not divisible by p_j for any j (to see this, reduce each multiplicand modulo p_j before multiplying). Thus $l(\mu)$ and $l(\lambda)$ have no factors in common, i.e., are relatively prime.

Finally, we must estimate the length $l(\mu)$. Clearly $p_i^{d_i} \leq l(\lambda) \leq \frac{(p+1)^2}{4}$ for each i. Also, $l(\lambda)$ cannot have more than $\log_2 l(\lambda)$ prime factors, so $k \leq \log_2 \frac{(p+1)^2}{4}$. Finally, each ν^i is a minimal (hence of length less than or equal to p) A-loop linked with itself at most $\frac{l(\lambda)}{2}$ times, so $l(\nu^i) = q_i \leq p \cdot \frac{(p+1)^2}{8} = \frac{p(p+1)^2}{8}$. Thus

$$l(\mu) = \prod_{i=1}^k (p_i^{d_i} + q_i) \leq \left(\frac{(p+1)^2}{4} + \frac{p(p+1)^2}{8}\right)^{\log_2 \frac{(p+1)^2}{4}}$$
$$= \left(\frac{(2+p)(p+1)^2}{8}\right)^{\log_2 \frac{(p+1)^2}{4}}.$$

$\square$

The converse to Theorems 3.23 and 3.27 is not true. The sofic shift associated to the matrix $M' = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ \hline -2 & 4 & 1 & 1 \\ -4 & 2 & 1 & 1 \end{array}\right)$ has positive entropy, but one can check that in this case $\bar{C}_M(X) = \bar{C}_{M'}(X)$.

Remark 3.32. It is easy to generalize the results in this section to matrices producing sofic shifts on three or more symbols. For details, see [12, §3.3].

4. **Applications.** Our first application of the results in the previous section is in the case when we begin with an adjacency matrix M' for a sofic shift. We can test whether the shift has positive entropy by computing $\bar{C}_{M'}(X)$, $\bar{C}_{M_1}(X)$, and $\bar{C}_{M_2}(X)$ (where M_i is the matrix for the restriction of the shift to the vertices labelled i) and applying Theorems 3.23 and 3.27.

A second application is to the case where we have two topological spaces X_1 and X_2 and a map f from their topological sum to itself. We produce symbolic

(a) M' (b) N'

FIGURE 6. Graphs corresponding to M' and N'

dynamics (a subshift of Σ_2^+) from f by associating to a point x its itinerary, i.e., the sequence of 1's and 2's reflecting the location of each iterate of x (we will make this more formal shortly). If M' is the matrix associated to a map on some dimension of homology induced by f, and M_1 and M_2 the obvious restrictions, then we would like to claim that the symbolic dynamics induced by f contains the sofic shift $(\Gamma_{M'}, \sigma_{M'})$ associated to M'. Unfortunately, this is not true, as the following example shows.

Example 4.1. Let $X_1 = S_1$, a single copy of S^1, and $X_2 = S_2 \vee S_3$, a wedge of two circles. Define the map f by sending S_1 to S_3 via an orientation-preserving homeomorphism, S_2 to S_1 via an orientation-preserving homeomorphism, and S_3 to a point $*$ of S_3. Then the only possible itineraries are $(1, 2, 2, \dots)$ and $(2, 2, \dots)$.

Let v_1^1 be a generator for $H_1(X_1)$, v_1^2 for $H_1(S_2)$, and v_2^2 for $H_1(S_3)$. With these basis elements, the matrix for the induced map on homology is $f_1 = M' = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)$, with corresponding graph in Figure 6(a). Then the sofic shift $(\Gamma_{M'}, \sigma_{M'})$ is trivial (there are no M'-paths of length greater than two).

But now let us choose a different basis. Let $\tilde{v}_1^1 = v_1^1$, $\tilde{v}_1^2 = v_1^2$, and $\tilde{v}_2^2 = v_1^2 + v_2^2$. Then the matrix N' for f_1 with this new basis is $\left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline -1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right)$, with corresponding graph in Figure 6(b). The associated sofic shift $(\Gamma_{N'}, \sigma_{N'})$ is nontrivial, and so is not a subshift of the symbolic dynamics associated to f.

Remark 4.2. This example seems to suggest that if we just pick the right basis, then the sofic shift associated to the homology matrix will be contained in the symbolic dynamics associated to the map. This is in fact true for simple spaces like the wedges of spheres in the example, but in general things seem to be more complicated.

A third application, and the original motivation for this paper, arises in using the discrete Conley index to detect symbolic dynamics in decompositions of isolated invariant sets.

The discrete Conley index is a powerful topological tool for studying isolated invariant sets of a given map f. Roughly speaking, it assigns to each such set a pointed space P and a base-point preserving map f_P, which is defined up to an equivalence relation. By studying the simpler map f_P we can draw conclusions about the original map f. Our discussion of the discrete Conley index is based on that in [6], where one can find more details and proofs of the theorems below. Let U be an open subset of a locally compact metric space X and suppose $f : U \to X$ is a continuous map.

Definition 4.3. For any set $N \subset U$ we define $\operatorname{Inv} N$, the *maximal invariant subset,* to be the set of $x \in N$ such that there exists an orbit $\{x_n\}_{n\in\mathbb{Z}} \subset N$ with $x_0 = x$ and $f(x_n) = x_{n+1}$ for all n. A compact set N is called an *isolating neighborhood* if $\operatorname{Inv} N \subset \operatorname{Int} N$. A set S is called an *isolated invariant set* if there exists an isolating neighborhood N with $S = \operatorname{Inv} N$. If N is an isolating neighborhood, we define the *exit set* of N to be

$$N^- := \{x \in N : f(x) \notin \operatorname{Int} N\}.$$

Definition 4.4. Let S be an isolated invariant set and suppose $L \subset N$ is a compact pair contained in the interior of the domain of f. The pair (N, L) is called a *filtration pair* for S provided N and L are each the closures of their interiors and

1. $\operatorname{Cl}(N \setminus L)$ is an isolating neighborhood of S,
2. L is a neighborhood of N^- in N, and
3. $f(L) \cap \operatorname{Cl}(N \setminus L) = \emptyset$.

Again, see [6] for proofs of the following theorems.

Theorem 4.5. *Let S be an isolated invariant set. For every neighborhood V of S, there exists a filtration pair (N, L) for S with $L \subset N \subset V$. Moreover there is a neighborhood of f in the C^0 topology such that for any $\tilde{f}$ in this neighborhood, $\tilde{S} = \operatorname{Inv}(N \setminus L, \tilde{f})$ is an isolated invariant set and (N, L) is a filtration pair for $\tilde{S}$.*

Theorem 4.6. *Let $P = (N, L)$ be a filtration pair for f and let N_L denote the quotient space N/L where the collapsed set L is denoted $[L]$ and is taken as the base-point. Then f induces a continuous base-point preserving map $f_P : N_L \to N_L$ with the property $[L] \subset \operatorname{Int} f_P^{-1}([L])$.*

Remark 4.7. Observe that we can identify the set $\operatorname{Inv}(N_L \setminus \{[L]\}, f_P)$ with $S = \operatorname{Inv}(\operatorname{Cl}(N\setminus L), f)$.

Given an isolated invariant set S, Theorem 4.5 tells us that we can find a filtration pair for it. Our choice of filtration pairs is not unique, even up to homotopy equivalence. Any two filtration pairs for S will, however, be shift equivalent (see [6] for definitions and proofs), which allows us to make the following definition.

Let S be an isolated invariant set, and consider the homotopy class of base-point preserving maps on N_L with f_P as a representative. We denote this collection $h_P(S)$. We may now define the Conley index.

Definition 4.8. Let S be an isolated invariant set for a continuous map f. Then define the *discrete homotopy Conley index* of S, $h(S)$, to be the shift equivalence class of $h_P(S)$, where $P = (N, L)$ is a filtration pair for S.

Assume that we have a continuous map of a locally compact metric space, $f : X \to X$, and an isolating neighborhood N. If N is the disjoint union of two other isolating neighborhoods N_1 and N_2, then we can associate a symbolic dynamical system to the map f restricted to $\operatorname{Inv} N$. We define (see [2]) a continuous map $\Theta : \operatorname{Inv} N \to \{1, 2\}$ by setting

$$\Theta(x) = \begin{cases} 1 & \text{if } x \in N_1, \\ 2 & \text{if } x \in N_2. \end{cases}$$

We relate the dynamics of f on $\operatorname{Inv} N$ to symbolic dynamics via the itinerary map $\rho : \operatorname{Inv} N \to \Sigma_2^+$ defined by

$$\rho(x) = (\Theta(x), \Theta(f(x)), \Theta(f^2(x)), \dots).$$

It is clear that ρ is continuous and that $\sigma \circ \rho = \rho \circ f$.

Following [10] and [2], we can use the discrete Conley index to detect interesting subshifts of the image shift $\rho(\operatorname{Inv} N)$, by analyzing the homology maps induced by f_P. (In all of what follows, we will be considering singular homology with real coefficients, so $H_*(-) = H_*(-;\mathbb{R})$.)

Let (K, L) be a filtration pair for $\operatorname{Inv} N$, and define K_i to be $K \cap N_i$ $(i = 1, 2)$, and similarly for L_i, so that the pointed space $K_L = (K_1)_{L_1} \vee (K_2)_{L_2}$.

Definition 4.9. Assume that for some positive integer q, $H_q(K_L, [L])$ is finite-dimensional, and that $\dim(H_q((K_i)_{L_i}, [L_i])) \geq 1$ for $i = 1, 2$. Let the set of vectors $(v_1^i, \dots, v_{n_i}^i)$ be a basis for $V_i := H_q((K_i)_{L_i}, [L_i])$. Then

$$(v_1^1 \oplus 0, \dots, v_{n_1}^1 \oplus 0, 0 \oplus v_1^2, \dots, 0 \oplus v_{n_2}^2)$$

is a basis for $V := H_q(K_L, [L])$. Let $M' = \left(\begin{array}{c|c} M_1 & A_{12} \\ \hline A_{21} & M_2 \end{array}\right)$ be the matrix representation in this basis for $(f_P)_q : H_q(K_L, [L]) \to H_q(K_L, [L])$, and let $M = \left(\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right)$. Define the matrices A_i $(i = 1, 2)$ by setting $A_1 = \left(\begin{array}{c|c} M_1 & 0 \\ \hline A_{21} & 0 \end{array}\right)$ and $A_2 = \left(\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & M_2 \end{array}\right)$.

Given a word $\omega = (\omega_0, \omega_1, \dots, \omega_k) \in \{1, 2\}^{k+1}$, define the matrix $(A_*)_\omega$ by setting

$$(A_*)_\omega := (A_{\omega_k} \circ A_{\omega_{k-1}} \circ A_{\omega_{k-2}} \circ \cdots \circ A_{\omega_1} \circ A_{\omega_0}).$$

For $i = 1, 2$, let k_i be a nonnegative integer and $\omega^i = (\omega_0^i, \dots, \omega_{k_i}^i)$ an element of $\{1, 2\}^{k_i+1}$. Also let $\alpha = (\alpha_0, \alpha_1, \dots)$ be an element of $\{1, 2\}^l$, where l is in $\mathbb{N} \cup \{+\infty\}$. Define the sequence $\alpha(\omega^1, \omega^2)$ (which will be infinite if α is infinite) by setting

$$\begin{aligned}\alpha(\omega^1, \omega^2) &:= ((\omega^{\alpha_0}), (\omega^{\alpha_1}), (\omega^{\alpha_2}), \dots) \\ &= (\omega_0^{\alpha_0}, \dots, \omega_{k_{\alpha_0}}^{\alpha_0}, \omega_0^{\alpha_1}, \dots, \omega_{k_{\alpha_1}}^{\alpha_1}, \omega_0^{\alpha_2}, \dots).\end{aligned}$$

Thus the word $\alpha(\omega^1, \omega^2)$ is a concatenation of the words ω^1 and ω^2, combined in the order specified by α.

The following theorem, which is essentially a corollary of [10, Th.4.2], is proven in [2, §4], using a slightly different but equivalent version of the Conley index. See also [13, Th. 4.13].

Theorem 4.10. *Suppose that ω^1 and ω^2 are two words such that $(A_*)_{\alpha(\omega^1,\omega^2)}$ is non-nilpotent for every finite α. Then for any $\beta = (\beta_0, \beta_1, \dots) \in \{1, 2\}^{+\infty}$ there exists a point $x \in S$ such that $\rho(x) = \beta(\omega^1, \omega^2)$.*

Remark 4.11. Since the map f_P depends on the choice of filtration pair for $\operatorname{Inv} N$, so do the characteristic polynomials $C_M(X)$ and $C_{M'}(X)$. But since any two choices lead to shift equivalent maps, $\bar{C}_M(X)$ and $\bar{C}_{M'}(X)$ are independent of the choice ([8, Prop. 7.3.7], and thus so are the following theorems.

Definition 4.12. We say that M and M' *satisfy Hypothesis H* if

1. $\bar{C}_M(X)$ does not divide $\bar{C}_{M'}(X)$, or
2. $\bar{C}_M(X)$ divides $\bar{C}_{M'}(X)$ but the polynomial $\bar{C}_{M'}(X)/\bar{C}_M(X)$ is not a product of cyclic polynomials.

Theorem 4.13 ([2, Thm. 1.1])**.** *If M and M' satisfy Hypothesis H, then there exists a positive integer d such that $f^d : S \to S$ factors onto the full two-shift.*

A disadvantage of this result is that it does not specify the power d. Thus it does not provide an estimate for the topological entropy of f. The following theorems will, with some additional hypotheses, provide a bound on d. Also, the proof of Theorem 4.13 is very algebraic. Our proofs may be more transparent to a dynamics based as they are on the theory of symbolic dynamics and adjacency matrices. This transparency allows us to generalize our results fairly easily.

On the other hand, the great advantage of Theorem 4.13 over our results is that it does not require additional hypotheses (such as the spaces having dimension one or the maps being cellular). Thus it can be applied in more general situations.

We would like to be able to claim that if there is an M'-loop with reduced itinerary $\omega \in \{1,2\}^k$, then the matrix $(A_*)_\omega$ is non-nilpotent, so that we could combine the results of the previous section with Theorem 4.10. However, we cannot make that claim in general, as the following example shows.

Example 4.14. Let $M' = \left(\begin{array}{c|c} M_1 & A_{12} \\ \hline A_{21} & M_2 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{array}\right)$. Then for any word ω, there is an M'-loop with ω as its reduced itinerary. However, $(A_*)_\omega$ is zero for any word ω of length greater than one, and hence nilpotent for any ω.

The problem is that the weights of the various M'-paths of a given length l can cancel each other out, leaving $(M')^l$ with zero entries. This cannot happen if M' is nonnegative, of course.

Proposition 4.15. *Assume that the matrix M' is nonnegative, and let ω be a finite word. If there is an M'-loop λ with $\bar{\psi}(\lambda) = \omega$, then the matrix $(A_*)_\omega$ is non-nilpotent.*

Proof. For any positive integer b and any p, $1 \leq p \leq n_1 + n_2$, $((A_*)^b_\omega)_{pp} = \sum_\mu w(\mu)$, where the sum is taken over all M'-loops μ at v_p such that $\bar{\psi}(\mu) = b \cdot \omega$.

Thus, if $\lambda_0 = v_p$, then

$$((A_*)^b_\omega)_{pp} = (w(\lambda))^b + \sum_\mu w(\mu),$$

where the sum is taken over all M'-loops μ at the corresponding vertex $v^{i(p)}_{j(p)}$ such that $\bar{\psi}(\mu) = b \cdot \omega$ and μ is not equal to λ linked with itself b times (the vertex corresponding to p is $v^{i(p)}_{j(p)}$, where $i(p) = 1$ if $p \leq n_1$ and 2 otherwise, and $j(p) = p - (1 - i(p))n_1$).

Since M' is nonnegative, every M'-loop has positive weight, so

$$((A_*)^b_\omega)_{pp} \geq (w(\lambda))^b > 0$$

for every b. Thus $(A_*)_\omega$ is non-nilpotent. □

As before, let $d = \max(\dim V_1, \dim V_2) = \max(n_1, n_2)$. Let b be the multiplicity of zero as a root of $C_M(X)$.

Corollary 4.16. *Assume that there exist bases for V_1 and V_2 such that the matrix M' representing $(f_P)_q$ with the induced basis on $V_1 \oplus V_2$ is nonnegative. Assume further that M and M' satisfy Hypothesis H. Then some power $k \leq \max((n_1 + n_2)d, B(b))$ of the map $f : S \to S$ factors onto the full two-shift. Thus $h_{top}(f) \geq \frac{\log 2}{k}$.*

Proof. This follows from Theorems 4.10, 3.23, and 3.27, and Proposition 4.15. □

This approach also works if V_1 and V_2 both have dimension one.

Theorem 4.17. *Assume that* $\dim V_1 = \dim V_2 = 1$. *If M and M' satisfy Hypothesis H, then there exists a positive integer $k \leq 2$ such that f^k factors onto the full two-shift.*

Proof. Since V_1 and V_2 each have only one vertex, an M'-path is uniquely determined by its itinerary. Thus if λ is an M'-loop with $\bar{\psi}(\lambda) = \omega$, then $(A_*)^b_\omega = (w(\lambda)b)$, so $(A_*)^b_\omega$ is non-zero.

Furthermore, in the case $n_1 = n_2 = 1$, all of the relevant loops in the proofs of Theorems 3.23 and 3.27 may be assumed to have length one or two. Thus we may proceed as in the proof of Corollary 4.16, with $k \leq 2$. □

Remark 4.18. The previous theorem can easily be proved directly by considering matrices of the form $\left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$, with a, b, c and d real numbers. The difficulty in higher dimensions comes from the fact that the space of $p \times p$ matrices is not a domain for $p > 1$.

To apply the results of the previous section in other circumstances, we will take a different approach from showing that $(A_*)_\omega$ is non-nilpotent. We will see that, under suitable hypotheses, we can verify directly that the sofic shift $(\Gamma_{M'}, \sigma_{M'})$ associated to M' is a subshift of $(\rho(S), \sigma|_{\rho(S)})$.

Theorem 4.19. *For $i = 1, 2$, let $(K_i)_{L_i} = X_i$ be a wedge of n_i q-spheres, so that $X_i = \bigvee_{j=1}^{n_i} S_{i,j}$. For each i and j, let $\bar{v}^i_j$ be a generator for $H_q(S_{i,j}) = \mathbb{R}$, and v^i_j the element of $H_q((K_i)_{L_i}, [L_i]) = \mathbb{R}^{n_i}$ induced by $\bar{v}^i_j$ under the inclusion $S_{i,j} \hookrightarrow X_i$. Then the set $\{v^i_j\}_{j=1}^{n_i}$ is a basis for $V_i = H_q((K_i)_{L_i}, [L_i])$. Let M and M' be the matrices for the maps induced by f_P on $H_q(K_L, [L]) = \mathbb{R}^{n_1+n_2}$ with these as basis elements. Then the sofic shift $(\Gamma_{M'}, \sigma_{M'})$ associated to M' is a subshift of $(\rho(S), \sigma|_{\rho(S)})$, so $h_{top}(f) \geq h_{top}(\sigma|_{\rho(S)}) \geq h_{top}(\sigma_{M'})$.*

Proof. Define a continuous map $\theta' : K_L \setminus [L] \to \{1, 2\}$ by setting $\theta'([x]) = i$, where $[x] \in (K_i)_{L_i}$.

Then define a map $\rho' : \mathrm{Inv}(K_L, f_P) \setminus \{[L]\} \to \Sigma_2^+$ by setting

$$\rho'([x]) = (\theta'([x]), \theta'(f_P([x])), \theta'({f_P}^2([x])), \dots).$$

It is clear that under the identification $[x] \longleftrightarrow x$ of $\mathrm{Inv}(N_L \setminus \{[L]\}, f_P)$ and $\mathrm{Inv}(\mathrm{Cl}(K \backslash L), f) = S$, $\rho(x) = \rho'([x])$. (See Remark 4.7.)

For any i_0, j_0, i_1, j_1 (where $i_k = 1, 2$ and $1 \leq j_k \leq n_{i_k}$, $k = 0, 1$), the map f_P on $K_L = X_1 \vee X_2$ induces a map $f^{i_1,j_1}_{i_0,j_0} : S_{i_1,j_1} \to S_{i_0,j_0}$ (defined by inclusion of S_{i_1,j_1}, followed by f_P, followed by projection to S_{i_0,j_0}). Then the $((1-i_0)n_1 + j_0, (1-i_0)n_1 + j_1)$-entry of M' is equal to $\pm \operatorname{degree}(f^{i_1,j_1}_{i_0,j_0})$ (see [5, §XI.4]). (We could make a convention to settle the question of the sign, if we cared, which we do not.) In particular, if this entry is not equal to zero, then $\operatorname{degree}(f^{i_1,j_1}_{i_0,j_0}) \neq 0$, so $f^{i_1,j_1}_{i_0,j_0}$ is a surjection ([5, Lemma XI.4.2]).

Let $\gamma = (\gamma_0, \gamma_1, \dots)$ be an infinite M'-path. If $\gamma_k = v^i_j$, define $b(\gamma_k)$ to be (i, j). Then the map $f^{b(\gamma_{k+1})}_{b(\gamma_k)} : S_{b(\gamma_{k+1})} \to S_{b(\gamma_k)}$ is surjective for any $k \geq 0$, by the definition of M'-path. Therefore the set $\bigcap_{k=0}^{\infty} {f_P}^{-k}(S_{b(\gamma_k)})$ is nonempty. We would like to say that for any $[x]$ in that set, $\rho'([x]) = \psi(\gamma)$, completing the proof, but that is not true, since the base point $[L]$ is in the set, and $\rho'([L])$ is not even defined.

So we need to be a little more careful. Recall from Theorem 4.6 that there is an open neighborhood U_0 of $[L]$ in K_L such that $f_P(U_0) = [L]$. For each $S_{i,j}$, let $\bar{S}_{i,j} = S_{i,j} \setminus U_0$. If $f_{i_0,j_0}^{i_1,j_1}$ is surjective, then in fact the image of $\bar{S}_{i_1,j_1}$ under $f_{i_0,j_0}^{i_1,j_1}$ is all of S_{i_0,j_0}. So if γ is an infinite M'-path, the set $S_\gamma := \bigcap_{k=0}^{\infty} {f_P}^{-k}(\bar{S}_{b(\gamma_k)})$ is nonempty. Clearly for any $[x]$ in S_γ, $\rho'([x]) = \psi(\gamma)$. Since $\rho(x) = \rho'([x])$, the theorem is proved. □

Remark 4.20. In order for Theorem 4.19 to work, we must take as the basis the set of generators for the qth homology of each q-sphere. The theorem's conclusion does not necessarily hold for other bases, as Example 4.1 shows.

In practice, we would probably not use the previous theorem (instead, we would assign each sphere its own label and study the resulting sofic shift on $n_1 + n_2$ symbols). It serves as a model for using the results from the previous section to bound the entropy of f, however, and as a simple example of the method of proof for the next theorem, which applies to the case that K_L is a finite cell complex and f_P a cellular map. Verifying the hypotheses on the characteristic polynomials is a little harder, however. We will need the following definitions.

Definition 4.21 ([7, Ch. 5])**.** Let $g : E \to E$ be a map of a compact ENR. The *homology zeta function* of g, Z_g, is the rational function defined by setting

$$Z_g(t) = \prod_{k=0}^{d} \det(I - g_k t)^{(-1)^{k+1}},$$

where $g_k : H_k(E) \to H_k(E)$ is induced by g, and d satisfies $H_k(E) = 0$ for $k > d$.

Theorem 4.22. *Assume that K_L is connected, that D is a finite cell decomposition for K_L, and that f_P is cellular with respect to D. Let b' be the maximum number of cells in any one dimension. If the rational fraction $\frac{Z_{f_P}(\frac{1}{X})}{Z_{f_{P_1}}(\frac{1}{X})Z_{f_{P_2}}(\frac{1}{X})}$ does not have the form $c(X-1)X^p\frac{P(X)}{Q(X)}$, where c is a constant, p is an integer, and P and Q are products of cyclic polynomials, then there exists a k, with $k \le \max(b'(b'-1), B(b'))$, such that $f^k : S \to S$ factors onto the full two-shift.*

Proof. For each $q \ge 0$, let $D^{(q)}$ be the q-skeleton of D, and let ${f_P}^{(q)}$ be the map induced by f_P on $D^{(q)}/D^{(q-1)}$. Let $C_q = H_q(D^{(q)}/D^{(q-1)})$. Then, with the appropriate differential, C_* is a chain complex with $H_*(C_*) = H_*(K_L)$ (see [1, §IV.10]). Let $\tau_q : C_q \to C_q$ be the chain map given by $\tau_q = ({f_P}^{(q)})_q : H_q(D^{(q)}, D^{(q-1)}) \to H_q(D^{(q)}, D^{(q-1)})$. Then the induced map $\tau_* : H_*(C_*) \to H_*(C_*)$ is the same as $(f_P)_* : H_*(K_L) \to H_*(K_L)$ (see [7, Ch. 4]). We have that $\prod_q Z_{{f_P}^{(q)}} = Z_{f_P}$ (see [7, Ch. 5]). Thus

$$\prod_q \det(I - \tau_q t)^{(-1)^{q+1}} = \prod_q \det(I - (f_P)_q t)^{(-1)^{q+1}}.$$

We can make a similar construction for the induced cell structure D_i on $(K_i)_{L_i}$. If $\tau_q^{(i)} = ({f_{P_i}}^{(q)})_q : H_q(D_i^{(q)}, D_i^{(q-1)}) \to H_q(D_i^{(q)}, D_i^{(q-1)})$, then, as above, we have that

$$\prod_q \det(I - \tau_q^{(i)} t)^{(-1)^{q+1}} = \prod_q \det(I - (f_{P_i})_q t)^{(-1)^{q+1}} = Z_{f_{P_i}}(t).$$

We wish to show that $\tau_q$and $\tau_q^{(1)} \oplus \tau_q^{(2)}$ satisfy Hypothesis H for some $q > 0$. Observe that

$$Z_{f_P}(t) = (1-t)^{-1} \prod_{q>0} \det(I - \tau_q t)^{(-1)^{q+1}}$$

and

$$Z_{f_{P_1}}(t) Z_{f_{P_2}}(t) = (1-t)^{-2} \prod_{q>0} (\det(I - \tau_q^{(1)} t) \det(I - \tau_q^{(2)} t))^{(-1)^{q+1}}.$$

Therefore $\frac{Z_{f_P}(\frac{1}{X})}{Z_{f_{P_1}}(\frac{1}{X}) Z_{f_{P_2}}(\frac{1}{X})} = c(X-1)X^p \prod_{q>0} \left(\frac{\bar{C}_{\tau_q}(X)}{\bar{C}_{\tau_q^{(1)}}(X) \bar{C}_{\tau_q^{(2)}}(X)} \right)^{(-1)^{q+1}}$, where c is a constant and p is an integer. Thus by hypothesis τ_q and $\tau_q^{(1)} \oplus \tau_q^{(2)}$ must satisfy Hypothesis H for some $q > 0$.

Now observe that $D_i^{(q)}/D_i^{(q-1)}$ is a wedge of q-spheres $\bigvee_{j=1}^{n_i} S_{i,j}$, one corresponding to each q-cell $E_{i,j}$ of D_i, and $D^{(q)}/D^{(q-1)} = (D_1^{(q)}/D_1^{(q-1)}) \vee (D_2^{(q)}/D_2^{(q-1)})$ (note that $n_1 + n_2 \leq b'$). Thus we may proceed as in the proof of Theorem 4.19. For any infinite τ-path γ, the set $\bigcap_{k=0}^{\infty} ({f_P}^{(q)})^{-k} (S_{b(\gamma_k)})$ is nonempty. Therefore, the set $E_\gamma := \bigcap_{k=0}^{\infty} {f_P}^{-k}(E_{b(\gamma_k)})$ is also nonempty, and, in fact, $E_\gamma \backslash \{[L]\}$ is nonempty as well (using the fact that $[L]$ is a superattractor, as in the proof of Theorem 4.19). Since $\rho'([x]) = \psi(\gamma)$ for any $[x] \in E_\gamma \backslash \{[L]\}$, the theorem is proved. □

Remark 4.23. As it was easy to generalize the results of the previous section to three or more pieces (Remark 3.32), so it is with the results in this section. Again, see [12, §3.3].

To the best of my knowledge, it is an open question whether the bounds on the entropy of f given in Theorem 4.19 always apply when the conditions of Theorem 4.13 are met, or whether we actually need the additional hypotheses of Theorem 4.19. An example showing that the latter case is the correct one should shed light on the relationship between homology matrices and adjacency matrices for sofic shifts.

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