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Noting the Difference: Musical Scales and Permutations

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Clough and Myerson's paper "Musical Scales and the Generalized Circle of Fifths" [**2**] and its companion paper in the *Journal of Music Theory* [**1**] are important and influential articles in the mathematical music theory literature. They deal with "the way the diatonic set (the white keys on the piano) is embedded in the chromatic scale (all the keys on the piano)" [**2**, p. 695] and give beautifully clear proofs of some elegant theorems. However, there is a slight error in some of the early results and proofs, which we point out and correct in this note. (For more background and detailed analysis of the material that follows, we highly recommend the original papers. We will follow the notation in [**2**] as closely as possible.)

Consider the two tone progressions A-B-C and C-D-E. They seem to have the same structure—each note is followed by the next note in the diatonic. However, if we play the two progressions, they sound quite different. This is because B and C are only one semitone apart (i.e., there is no black key between them on the keyboard; see Figure 1), whereas D and E, the corresponding pair in the second progression, are two semitones apart (there is a black key between them). Roughly stated, the question that Clough

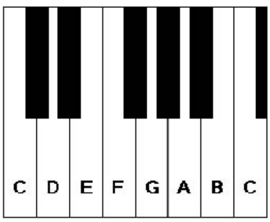

Figure 1. A piano keyboard.

and Myerson address is how many different-sounding progressions there are among those with the same structure. We now make this more precise.

We define the *diatonic set* (or *scale*) to be {A, B, C, D, E, F, G}, the set of notes corresponding to the white keys on a piano. (We are really considering equivalence classes of notes under translation by an octave, that is, A represents A2 (110 Hz), A3 (220 Hz), etc. Furthermore, this is really just one example of a diatonic set; there are many others embedded in the chromatic scale, and the results in this paper apply to any of them.)

A *k-note line* ($1 \leq k \leq 7$) is an ordered sequence of k distinct elements of the diatonic set, denoted X_1-X_2-$\cdots$-X_k. The shortest nontrivial line is a two-note line, called an *interval*. An interval has both a diatonic length and a chromatic length (in fact, there is a third notion of length, measured in fifths, that we will use later). The *diatonic length* of X_1-X_2, denoted $D(X_1\text{-}X_2)$, is the number of steps in the diatonic needed to go from the first note in the interval to the second, where the direction of movement is always "up" the scale (thus, the diatonic length of A-C is two). Given a line X_1-X_2-$\cdots$-X_k, we define its *diatonic length vector* to be $(D(X_1\text{-}X_2), D(X_2\text{-}X_3), \ldots, D(X_{k-1}\text{-}X_k), D(X_k\text{-}X_1))$.

We define the *chromatic set* (or *scale*) to be {A, A#, B, C, C#, D, D#, E, F, F#, G, G#}, the set of notes corresponding to all the keys on a piano. The *chromatic length* of $X_1 - X_2$, denoted $|X_1\text{-}X_2|$, is the number of steps in the chromatic needed to go from the first note to the second, where again the direction of movement is always up the scale (so the chromatic length of A-C is three). In other words, the chromatic length is the length measured in semitones. We define the *chromatic length vector* of the line X_1-X_2-$\cdots$-X_k to be $(|X_1\text{-}X_2|, |X_2\text{-}X_3|, \ldots, |X_k\text{-}X_1|)$.

We use these notions of length to define two equivalence relations on the set of lines. First, we say that two k-note lines are in the same *genus* if they have the same diatonic length vector. For example, the genus of A-C-D, denoted ⟨A-C-D⟩, is the set

{A-C-D, B-D-E, C-E-F, D-F-G, E-G-A, F-A-B, G-B-C}.

Next, we define a stronger equivalence relation by saying that two k-note lines in the same genus are in the same *species* if they have the same chromatic length vector as well. Thus, the genus ⟨A-C-D⟩ is partitioned into three species, {A-C-D, B-D-E, D-F-G, E-G-A}, {C-E-F, G-B-E}, and {F-A-B}, with chromatic length vectors (3, 2, 7), (4, 1, 7), and (4, 2, 6), respectively. Lines in the same species sound the same when we play them, while lines in different species sound different.

A final tool that we need is the *circle of fifths* (see Figure 2). We construct the circle of fifths by starting anywhere, say at F, and moving to the note a distance of seven semitones (called a *fifth*) away. We continue until we have reached every note in the chromatic and ended up back at F. In Figure 2, the numbers on the inside of the circle are the diatonic distances between consecutive notes, while those on the outside are the chromatic distances. Since we are interested only in diatonic notes, we consider only the top portion of the circle of fifths (consisting of the diatonic notes with an edge of chromatic length six added from B to F), which we will call the *diatonic circle*. Observe that we can represent any line graphically on the diatonic circle, as in Figure 3.

We observe that, since chromatic lengths are defined modulo 12, we can use the circle of fifths to find them. For example, to find |F-G|, we look at the circle of fifths and find that $|\text{F-C}| = 7$ and $|\text{C-G}| = 7$, so $|\text{F-G}| = 7 + 7 \pmod{12} = 2$.

Note that A-C-D has three elements, and its genus contains three species. This is not a coincidence; in fact, it is Theorem 1 of [**2**]. However, in proving that result,

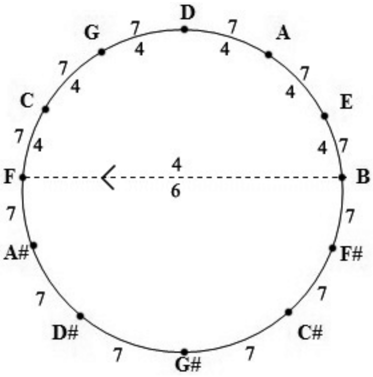

Figure 2. The circle of fifths.

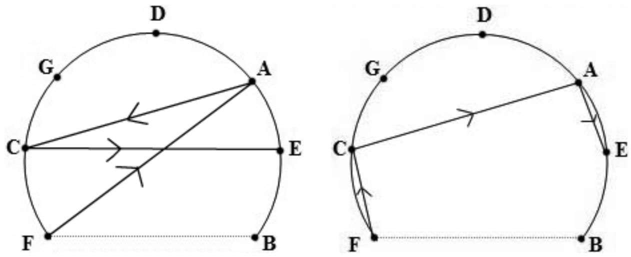

Figure 3. The lines F-A-C-E and F-C-A-E.

the authors implicitly assume that every line is in clockwise order around the circle of fifths, which is clearly not true (for example, F-C-A-E is in clockwise order, but F-A-C-E is not). Thus the proof of Theorem 1 and the statement of Corollary 1 are incorrect. We now modify them slightly to correct them.

Theorem 1. *For any given k* ($1 \leq k \leq 7$) *and any k-note line the genus containing that line comprises exactly k species.*

Proof. We first give Clough and Myerson's proof, which applies in the case that the line is in clockwise order. Given a line X_1-X_2-$\cdots$-X_k, we obtain the other lines in its genus by cycling around the diatonic circle. Since B-F is the only interval on the diatonic circle that does not have chromatic length seven, it is the location of this interval in a line that determines the line's species. There are k possible locations (between X_1 and $X_2, \ldots, X_{k-1}$ and X_k, or X_k and X_1), hence k different species.

We now consider the general case, when the line is not necessarily in clockwise order. Given a line X_1-X_2-$\cdots$-X_k, let π be one of the k permutations of $\{1, \ldots, k\}$ that puts it in order (i.e., the line $X_{\pi(1)}$-$X_{\pi(2)}$-$\ldots$-$X_{\pi(k)}$ is in clockwise order). It is clear that the genus of the permuted line consists of the images of the lines in the original's genus under the same permutation π. We claim that the genera $\langle X_1\text{-}X_2\text{-}\cdots\text{-}X_k\rangle$ and $\langle X_{\pi(1)}\text{-}X_{\pi(2)}\text{-}\cdots\text{-}X_{\pi(k)}\rangle$ contain the same number of species.

To see this, observe that, for any l and m, $|X_{\pi(l)}\text{-}X_{\pi(m)}|$ is equal to

$$|X_{\pi(l)}\text{-}X_{\pi(l)+1}| + |X_{\pi(l)+1}\text{-}X_{\pi(l)+2}| + \cdots + |X_{\pi(m)-1}\text{-}X_{\pi(m)}|$$

modulo 12 (we define X_0 to be X_k if $\pi(l) > \pi(m)$). For example, consider the line F-A-C-E. We can permute it to get the line F-C-A-E, which is in clockwise order. Then we have |F-C| = |F-A| + |A-C| = 4 + 3 = 7, as it should be.

Thus a line's chromatic length vector determines the chromatic length vector of a permutation of that line. Since any permutation is invertible, the reverse is also true—the chromatic length vector of the permuted line determines the chromatic length vector of the original. It follows that there is a one-to-one correspondence between the set of distinct chromatic length vectors of the original line and the set of those for the permuted line. In other words, $\langle X_1\text{-}X_2\text{-}\cdots\text{-}X_k\rangle$ has the same number of species as $\langle X_{\pi(1)}\text{-}X_{\pi(k)}\text{-}\cdots\text{-}X_{\pi(k)}\rangle$. Accordingly, a k-note line has the same number of species as any of its clockwise reorderings, and Clough and Myerson's proof tells us that a clockwise line has k species. ■

Example 1. The line F-A-C-E should have four species, as in fact it does. They are {F-A-C-E, C-E-G-B}, {G-B-D-F}, {A-C-E-G, D-F-A-C, E-G-B-D}, and {B-D-F-A}, with chromatic length vectors $(4, 3, 4, 1)$, $(4, 3, 3, 2)$, $(3, 4, 3, 2)$, and $(3, 3, 4, 2)$, respectively.

We are now able to state a corrected version of Corollary 1 from [**2**], the proof of which follows immediately from the foregoing proof.

Corollary 1. *If a given line is in clockwise order, then the numbers of lines in the species contained in the genus of that line are given by the distances between each note in the line and the next note in the line. Distances are measured in fifths in the sense of clockwise travel around the diatonic circle, and they include the distance from the last note to the first. In general, the numbers of lines in the species contained in the genus of a given line are the same as those for any clockwise reordering of the line.*

Example 2. Again, we consider the line F-A-C-E from Example 1. The line F-C-A-E is a clockwise reordering. Since C is one step away from F on the circle of fifths, A three steps from C, E one step from A, and F two steps from E, we expect the genus ⟨F-A-C-E⟩ to have two species with a single element, one species with two elements, and one species with three elements, which is exactly what we found in Example 1.

Clough and Myerson prove corresponding theorems for more general kinds of scales than the usual chromatic and diatonic; these contain essentially the same slight errors and can be corrected in the same way. In addition, we have seen that the genus of a line and any of its permutations have the same structure of species. Thus we may talk about the structure of *chords*, which are unordered subsets of the diatonic (e.g., ABC and CBA represent the same chord). We leave the details to the interested reader.

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