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Approval voting in subset elections

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Summary. Approval voting is designed to be "insensitive to numbers" of voters, and likely to elect a Condorcet candidate. However, the result of an election among one group of candidates gives no information about the results of elections among any other groups, even if every voter follows the recommended utility-maximizing strategy, which places strong restrictions on the individual voter's subset ballots. Thus the addition of a single candidate could completely reverse the outcome of an election, or a Condorcet candidate could finish last.

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1 Introduction

Approval voting, where voters choose their ballots based on a utility-maximizing strategy given in [1], is a system designed to avoid many of the flaws of other voting systems, such as plurality or preferential voting, in the case of multicandidate elections. In particular, it is claimed, among other things, that it "better reveals the overall acceptability of the candidates independent of the field in which they run" ([1], p. 5), that it is "insensitive to numbers" of voters ([1], p. 8), and that it has "a strong propensity to elect ... Condorcet candidates" ([1], p. 10). (A candidate is a Condorcet candidate if he wins all pairwise majority vote elections against all other candidates.) Thus, if under approval voting the outcome for a contest involving candidates A, B, C, and D is $A \succ B \succ C \succ D$, one would like to be confident that an election involving only A, B, and C would not have as its result $C \succ B \succ A$. A result asserting that approval voting can make no such guarantee was essentially proven in [6]. However, as was pointed out in [2] and [3], this result does not assume that the voters are using the prescribed utility-maximizing strategy. Thus one could hope that this further assumption, which places restrictions on an individual voter's subset ballots, might impose some order on the subset election results. Unfortunately, it does not.

Theorem 1 *Under approval voting, given the overall election results, there are no restrictions on the subset election results, even if every voter follows the recommended utility-maximizing strategy. In other words, information about the results of an election among one group of candidates gives no information about the results of elections among any other groups.*

Thus, for example, a Condorcet winner could come in last in the overall election. Or, the addition or subtraction of a single candidate from the field could completely reverse the outcome of the election. This result is more surprising when one considers that the strategy an approval voter uses in selecting his ballot places rather strong restrictions on the individual voter's ballot for subset elections. We now describe that strategy and explain the restrictions it places on individual subset election ballots. (For the rest of the paper, when we say "approval voting" we will mean "approval voting with every voter using the utility-maximizing voting strategy.")

2 Mean-utility strategy and assumptions

Under the mean utility strategy recommended in [1], a voter approves of a given candidate only if the utility that the voter would derive from that candidate's election is greater than the average utility over all the candidates running in the election. More formally, let $\mathscr{C} = \{C_1, \ldots, C_n\}$ be the set of all n candidates in a given election. Let $\{C_{i_1}, \ldots, C_{i_m}\}$ (with $1 \leq i_1 < i_2 < \ldots < i_m \leq n$) be a subset of $\mathscr{C}$. If u is a given voter's utility function, define that voter's *mean utility function* $\bar{u}$ by

$$\bar{u}(\{C_{i_1}, \ldots, C_{i_m}\}) = \frac{\sum_{j=1}^{m} u(C_{i_j})}{m}. \tag{1}$$

Define a voter's *individual approval function* A_{i_j} by

$$A_{i_j}(\{C_{i_1}, \ldots, C_{i_m}\}) = \begin{cases} 1, & \text{if } u(C_{i_j}) > \bar{u}(\{C_{i_1}, \ldots, C_{i_m}\}) \\ 0, & \text{if } u(C_{i_j}) < \bar{u}(\{C_{i_1}, \ldots, C_{i_m}\}) \end{cases}. \tag{2}$$

Thus $A_{i_j}(\{C_{i_1}, \ldots, C_{i_m}\})$ reflects whether or not the given voter would approve of candidate C_{i_j} when compared with the group $\{C_{i_1}, \ldots, C_{i_m}\}$. (So if a voter is voting on the subset of candidates $\mathscr{S} = \{C_{i_1}, \ldots, C_{i_m}\}$, his ballot would be $(A_{i_1}(\mathscr{S}), \ldots, A_{i_m}(\mathscr{S}))$). Finally, we say that $C_{i_j} \succ_{\mathscr{S}} C_{i_k}$ (C_{i_j} *is preferred to* C_{i_k} *in subset* $\mathscr{S}$) if

$$A_{i_j}(\mathscr{S}) = 1 \text{ and } A_{i_k}(\mathscr{S}) = 0 \tag{3}$$

(i.e., if $u(C_{i_j}) > \bar{u}(\mathscr{S})$ and $u(C_{i_k}) < \bar{u}(\mathscr{S})$). Similarly, we say $C_{i_j} \succeq_{\mathscr{S}} C_{i_k}$ if $A_{i_j}(\mathscr{S}) \geq A_{i_k}(\mathscr{S})$.

Because of the form of the individual approval function, there are restrictions on what subset rankings an individual voter can have. We will assume that there are no ties (i.e., $u(C_j) \neq u(C_k)$ if $j \neq k$), since a tie means that the voter must make arbitrary choices, thus weakening any possible connection among ballots for different elections. With this assumption, in every subset of candidates there must be at least one candidate above average for the subset and at least one below average. Thus our first restriction is

Individual Restriction 1 *For each subset* $\mathscr{S} = \{C_{i_1}, \ldots, C_{i_m}\}$ *of* $\mathscr{C}$ *(with* $m \geq 2$*), there exist integers* j *and* k*, with* $1 \leq j, k \leq m$*, such that* $A_{i_j}(\mathscr{S}) = 1$ *and* $A_{i_k}(\mathscr{S}) = 0$.

Second, note that if $C_j \succ_{\mathscr{S}} C_k$ for any subset of candidates $\mathscr{S}$, then $u(C_j) > u(C_k)$. Thus C_k could never be preferred to C_j in any subset (although the voter could be indifferent between C_j and C_k in some subsets). This gives us

Individual Restriction 2 *If* $C_j \succ_{\mathscr{S}} C_k$ *for any* $\mathscr{S} \subset \mathscr{C}$*, then* $C_j \succeq_{\mathscr{T}} C_k$ *for every* $\mathscr{T} \subset \mathscr{C}$ *such that* $C_j, C_k \in \mathscr{T}$.

Finally, notice that if a better-than-average candidate is removed from a subset of candidates, the average utility of the set of remaining candidates will be lower than the average utility of the original subset. Thus if one of the remaining candidates was better than average originally, he will still be better than average in the subset. So we have

Individual Restriction 3 *If* $A_{i_j}(\{C_{i_1}, \ldots, C_{i_m}\}) = 1$, *then* $A_{i_l}(\{C_{i_1}, \ldots, \widehat{C_{i_j}}, \ldots, C_{i_m}\}) \geq A_{i_l}(\{C_{i_1}, \ldots, C_{i_m}\})$ *for all* $l \neq j$*. Similarly, if* $A_{i_k}(\{C_{i_1}, \ldots, C_{i_m}\}) = 0$, *then* $A_{i_l}(\{C_{i_1}, \ldots, \widehat{C_{i_k}}, \ldots, C_{i_m}\}) \leq A_{i_l}(\{C_{i_1}, \ldots, C_{i_m}\})$ *for all* $l \neq k$.

As an example, assume that a voter's ballot for the set of candidates $\mathscr{C} = \{C_1, C_2, C_3\}$ is $(1, 1, 0)$. Then his ballot for the subset $\mathscr{S} = \{C_1, C_3\}$ could not be $(0, 1)$ (by Individual Restriction 2, the fact that $C_1 \succ_{\mathscr{C}} C_3$ implies that $C_1 \succeq_{\mathscr{S}} C_3$, and by Individual Restriction 3, the fact that $A_1(\mathscr{C}) = 1$ implies that $A_1(\mathscr{S}) = 1$).

Thus there are rather severe limitations on the possible subset rankings of a group of candidates for an individual voter. In contrast, Theorem 1 says that there are *no* restrictions on the subset rankings for the entire electorate. In the example above, the election results could be $C_1 \succ C_2 \succ C_3$ for an election comparing all three candidates, but $C_2 \succ C_1$, $C_3 \succ C_1$, and $C_3 \succ C_2$ for the pairwise elections. Thus the Condorcet winner, C_3, finishes last in the overall election.

3 Proof

Theorem 1 is actually a corollary of the following theorem of Saari's.

Theorem 2 ([4]) *Under the plurality voting system, where each voter votes for his or her top-ranked candidate, given the overall election results, there are no restrictions on the subset election results.*

Theorem 1 follows, since for certain voter profiles, the plurality tally and the approval tally will coincide. (This will occur if, for every voter, the following condition holds: if C_{i_j} is the voter's jth ranked candidate, then $u(C_{i_j}) > \sum_{k=j+1}^{n} u(C_{i_k})$ for all $1 \leq j \leq n$.)

Saari's proof uses linear algebra, and reduces the problem to showing that a certain matrix has maximal rank. Another proof, which allows one to actually construct examples, is as follows. The idea is to successively add to the electorate groups of voters which will change the rankings for all subsets of m candidates without changing the rankings of the subsets of p candidates for any $p > m$. To do this, we will take advantage of some of the symmetries inherent in the approval voting method.

First, we give an example to make clear the idea of the proof. We will produce the outcome in the example above, in which we have a three candidate election ($\mathscr{C} = \{C_1, C_2, C_3\}$) and the full election outcome is $C_1 \succ C_2 \succ C_3$, but the pairwise outcomes are $C_2 \succ C_1$, $C_3 \succ C_1$, and $C_3 \succ C_2$. Begin by selecting any group of voters which gives the $C_1 \succ C_2 \succ C_3$ three-way outcome, without worrying about the resulting pairwise results. Next, form a group of three voters, call it V, consisting of one voter v_1 whose three-way ballot is $(1, 0, 0)$, one, v_2, whose ballot is $(0, 1, 0)$, and one, v_3, whose ballot is $(0, 0, 1)$. Now, v_1's ballots for the $\{C_1, C_2\}$ and $\{C_1, C_3\}$ pairwise elections are forced on him by the individual restrictions (they both must be $(1, 0)$), but his $\{C_2, C_3\}$ ballot can be anything we choose (either $(1, 0)$) or $(0, 1)$). Since we want the outcome for this pairwise election to be $C_3 \succ C_2$, we choose $(0, 1)$.

Similarly, v_2's $\{C_1, C_2\}$ and $\{C_2, C_3\}$ pairwise ballots are forced on him, but we are free to choose his $\{C_1, C_3\}$ to be $(0, 1)$ to support the outcome $C_3 \succ C_1$. Finally, we have no choice for v_3's $\{C_1, C_3\}$ or $\{C_2, C_3\}$ ballots, but we can choose his $\{C_1, C_2\}$ ballot as $(0, 1)$.

Now combine this new group V with our original group of voters. The three-way outcome is the same as in the original group, since v_1, v_2, and v_3's three-way ballots add up to give $(1, 0, 0)+(0, 1, 0)+(0, 0, 1) = (1, 1, 1)$, a complete tie, which has no effect on the outcome. The pairwise outcomes are changed, however. Consider the $\{C_1, C_2\}$ outcome. Voter v_1's $(1, 0)$ ballot cancels out v_2's $(0, 1)$ ballot, so the outcome for the combined groups moves from the original outcome toward the $C_2 \succ C_1$ outcome reflected by v_3's $(0, 1)$ ballot. Similarly, the $\{C_1, C_3\}$ and $\{C_2, C_3\}$ pairwise election outcomes move toward $C_3 \succ C_1$ and $C_3 \succ C_2$ respectively. If we add enough copies of the voter group V, the pairwise results of the combined group will be as desired. Thus we were able to adjust all of the subset election outcomes to what we wanted.

Proof (of Theorem 1) . The idea for producing arbitrary subset results for an n-candidate election ($\mathscr{C} = \{C_1, \dots, C_n\}$) is similar, although the details are more complicated. We repeat the steps above inductively. First, we pick any group of voters that gives the desired n-way outcome. Next, assuming that we have a group that gives the desired results for all n-, $(n-1)$-,...,$(n-m+1)$-way elections, we add a group of voters to get the desired $(n-m)$-way election results.

First, we form a group V_{n-m} of $\binom{n}{m}$ voters, each of whom approves of m different candidates in the full n-way election (so each one's n-way ballot consists of m 1's and $n-m$ 0's, and there is one voter in V_{n-m} for each of the $\binom{n}{m}$ ways to construct such a ballot). By symmetry, the n-way election result for V_{n-m} will be a complete tie, so adding V_{n-m} to our original group will not change the original n-way outcome.

Next we need to determine the ballots for V_{n-m} for the $(n-1)$-,..., $(n-m)$-way elections. (We can choose the ballots for the smaller subset elections arbitrarily, as we will fix the outcomes of those elections later.) Assume that a voter v_i in the group V_{n-m} approves of the set of candidates $\{C_{i_1}, \dots, C_{i_m}\}$ in the n-way election. This ballot gives no information about (and thus puts no restrictions on) v_i's ballot for the $(n-m)$-way election involving $\mathscr{C} - \{C_{i_1}, \dots, C_{i_m}\}$. Thus we want to choose this ballot to give the desired outcome for the $\mathscr{C} - \{C_{i_1}, \dots, C_{i_m}\}$ election.

We still need to choose v_i's ballots for the remaining $(\binom{n}{m} - 1)$ $(n-m)$-way elections, as well as for all the $(n-1)$-,..., $(n-m+1)$-way elections, in such a way that our choices do not change the outcome of these elections or restrict our choices for the $\mathscr{C} - \{C_{i_1}, \dots, C_{i_m}\}$ ballot. Ideally, these ballots would reflect only what was on v_i's n-way ballot. That is, v_i would approve of candidate C_j in a k-way election if and only if he approved of C_j in the full n-way election. (Thus v_i's ballot for the $\{C_{i_1}, \dots, C_{i_k}\}$ election ($k \geq n-m$) would be $(A_{i_1}(\mathscr{C}), \dots, A_{i_k}(\mathscr{C}))$. The advantage of this method is that we add no new information or restrictions by simply copying the information we already had from the n-way ballots. Choosing the ballots this way will clearly not violate Individual Restrictions 2 or 3. Unfortunately, it may violate Restriction 1, by giving a ballot either of all 1's or all 0's.

The only way this could violate Restriction 1 by giving a ballot consisting of all 0's for an election of $(n-m)$ or more candidates is if the candidates are $\mathscr{C} - \{C_{i_1}, \dots, C_{i_m}\}$. In this case, we take advantage of the situation to produce the outcome that we want, as described above. If $m < n/2$, this is the only problem with Restriction 1, and we can construct our group V_{n-m} exactly as above. This is because the number of 1's in the n-way ballot is m. Thus we could not get a ballot of all 1's until we started to look at the m-way elections. But since $n-m > n/2 > m$, we do not have to worry about the m-way elections yet, as we will fix their outcomes later. Thus we can choose any ballot we want that is consistent with the individual restrictions.

In fact, Restriction 1 is not a problem if $m = n/2$. In this case, $n - m = m$, so the first ballot of all 1's would occur in the $(n - m)$-way election among the candidates $\{C_{i_1}, \ldots, C_{i_m}\}$. Since this is one of the outcomes that we want to fix, and we can pick any ballot we want for this election, we simply pick the ballot giving the desired outcome. Thus the method (picking the outcomes we want for the $(n - m)$-way elections, the outcomes corresponding to the full n-way ballot for the $(n - 1)$-,...,$(n - m + 1)$-way elections, and anything consistent with the restrictions for the $(n - m - 1)$-,...,2-way elections) will work to fix the outcomes for all m-way elections without changing the outcomes for the elections with more candidates, as long as $m \leq n/2$. In particular, if the number n of candidates is four or smaller, then the above method will work to produce the group V_{n-m} for all meaningful $(n - m)$-way elections (4-way, 3-way, and pairwise).

However, if we are trying to change the outcome of an $(n - m)$-way election where $m > n/2$, we will have a problem with the method. We could get a ballot of all 1's for an m-way election. Since $n - m < n/2 < m$, we cannot change the outcome of this election, and thus we cannot simply pick any ballot we want. Note, though, that any election that would get a ballot of all 1's would be an election among the candidates in a subset of $\{C_{i_1}, \ldots, C_{i_m}\}$, which subset would be disjoint from $\mathscr{C} - \{C_{i_1}, \ldots, C_{i_m}\}$, so that our choice of ballots for the prior election would place no restrictions on our choice of ballots for the latter. Thus the only potential danger in our choice of ballots for the m-way election is changing the outcome of that election, not restricting our choices for the $\mathscr{C} - \{C_{i_1}, \ldots, C_{i_m}\}$ election.

Assume that a voter v_i in the group V_{n-m} approves of the set of candidates $\mathscr{S}_i = \{C_{i_1}, \ldots, C_{i_m}\}$ in the n-way election. Then he would have a ballot of all 1's for the election among the subset of candidates $\mathscr{S}_i$, which is not an allowable ballot. So we must make another choice. But the only way to keep his ballot from changing the outcome of the $\mathscr{S}_i$ election is for the ballot to be a complete tie (i.e., either all 0's or all 1's), which is not an option. A solution is to break v_i up into m different voters, $v_i^1, \ldots, v_i^m$, whose ballots agree with those of the original v_i for the n-, $(n - 1)$-, ..., $(m + 1)$-way elections and for all m-way elections except the one involving $\mathscr{S}_i$. Each v_i^j approves of a different one of the m candidates in $\mathscr{S}_i$ and disapproves of the rest. The $\mathscr{S}_i$ ballot for each v_i^j is allowable (it consists of one 1 and $m - 1$ 0's), and they add up to cancel each other out and give a complete tie as their outcome, thus not changing the original outcome.

For each v_i^j, we can choose his ballots for elections among smaller subsets of candidates by simply copying the information from his m-way ballots. If $m - 1 > n - m$, then we run into the same problem again. Each v_i^j will have one $(m - 1)$-way election for which his ballot would be all 0's (the election among the $m - 1$ candidates from $\{C_{i_1}, \ldots, C_{i_m}\}$ of whom he did not approve). The solution is the same: split v_i^j up into $m - 1$ voters, each of whom approves of one of the $m - 1$ candidates in question and disapproves of the rest. All of these voters add up to give a complete tie, so they do not change the outcome of the election. We can continue in this way, splitting v_i into more and more voters, until we get to the $(n - m)$-way elections.

So we can construct a group V_{n-m} of voters in the above manner, for each m, $0 < m < n - 1$. If we add enough copies of V_{n-1} to our original group of voters, then construct and add enough copies of V_{n-2}, and continue in this way until we have constructed and added enough copies of V_2, the resulting group of voters will give the desired outcome for each election among each subset of candidates.

□

Note that this method is not the only way of constructing such a set of voters, nor is it likely to be the most efficient. Given any particular example, one could probably make intelligent, rather than arbitrary, choices for many of the ballots, thus reducing the size of the final group of voters. The above method will work, however, for any election using approval voting.

4 Conclusion

It is perhaps not too surprising that the individual restrictions created by the use of the mean-utility strategy are not enough to prevent contradictory results in subset elections under approval voting. For example, any positional voting system where differently ranked candidates get different tallies on a voter's ballot places stronger restrictions on the individual voter (in this case a voter's all-candidate ballot completely determines his ballot for all subset elections). Yet Theorem 1 holds for almost any positional system (although not, in particular, the Borda count) ([5]). Thus approval voting is hardly the only voting system to experience inconsistent subset election results. However, since approval voting does have this flaw, it is not clear that it is superior to other systems that do not.

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