Key

Midterm

Math 6C

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Instructions: This is a closed book exam. No calculators are allowed. Show all work and justify all answers.

15 points 1. You have a business in which you have a choice as to how many hours you work each day, (x), and how much you spend on operating costs in dollars per day, (y). Your production, as a function f of x and y is shown below:

$y \setminus x$	6	6.5	7	7.5
20	971	1010	1098	1110
21	990	1029	1086	1132
22	1009	1049	1107	1153

a) What does $f(7, 21)$ represent in this scenario?

If I work 7 hrs a day and spend \$21 per day on costs, then I can produce 1086 objects.

b) Estimate $f_x(7, 21)$. $\approx \dfrac{f(7+\frac{1}{2}, 21) - f(7,21)}{\frac{1}{2}}$

$$= \frac{1132 - 1086}{\frac{1}{2}}$$

$$= \frac{46}{\frac{1}{2}} = 92$$

c) What does $f_x(7, 21)$ represent in this scenario? When I'm working 7 hrs a day and spending \$21 per day on cost, if I put in an xtra ~~[illegible]~~ hr, can increase production at a rate of ~~[illegible]~~ 92

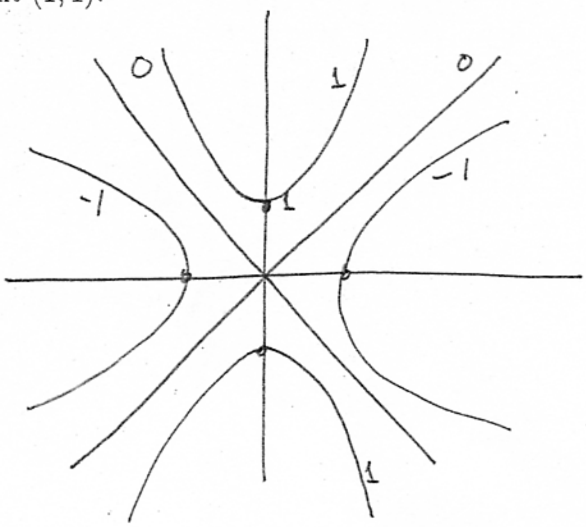
15 points

2. Draw some level curves for the function $z = y^2 - x^2$. Include the level curve that goes through the point $(1,1)$.

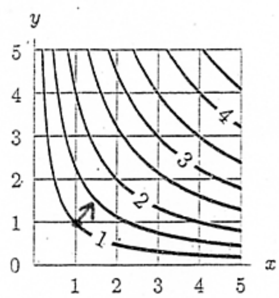
$0 = y^2 - x^2$ has $x^2 = y^2$

$1 = y^2 - x^2$ has $y^2 = 1 + x^2$

$-1 = y^2 - x^2$ has $y^2 = x^2 - 1$



b) In the picture shown below, draw in the gradient of the function at the point $(1,1)$.



c) Using ~~your~~ the picture above, is the directional derivative of the function at the point $(1,1)$ in the direction of $\vec{i} + 2\vec{j}$ positive or negative? Explain.

Positive; function increases in the direction.

12 points

3. Find an equation for the tangent plane to $z^2 - 2x^2 - 2y^2 = 12$ at the point $(1, -1, 4)$. Use your tangent plane to estimate the value of z at $(x, y) = (1.5, 0.5)$.

$$z^2 = 12 + 2x^2 + 2y^2$$

$$z = \sqrt{12 + 2x^2 + 2y^2}$$

$$z_x = \tfrac{1}{2}(12 + 2x^2 + 2y^2)^{-1/2} \cdot 4x = \frac{2x}{\sqrt{12+2x^2+2y^2}}$$

$$z_y = \tfrac{1}{2}(12 + 2x^2 + 2y^2)^{-1/2} \cdot 4y = \frac{2y}{\sqrt{12+2x^2+2y^2}}$$

So $z_x(1,-1) = \dfrac{2}{\sqrt{12+2+2}} = \dfrac{2}{\sqrt{16}} = \dfrac{2}{4} = 1/2$

$z_y(1,-1) = \dfrac{-2}{\sqrt{16}} = \dfrac{-2}{4} = -1/2$

So $z = 4 + 1/2(x-1) - \frac{1}{2}(y+1)$

So at $(1.5, 0.5)$,

$$z \approx 4 + 1/2(1/2) - 1/2(3/2) = 4 + 1/4 - 3/4 = 4 - 1/2$$
$$= 3\tfrac{1}{2}.$$

22 points

4. A hot metal plate is situated on an xy-plane such that the temperature $T(x,y) = k\frac{1}{\sqrt{x^2+y}}$. If the temperature at $(3,7)$ is 10, find k.

$$10 = k \frac{1}{\sqrt{9+7}}$$

$$10 = k \frac{1}{\sqrt{16}} \qquad \text{so } k = 40$$

$$10 = \frac{k}{4}$$

b) Find the rate of change of T at $(2,5)$ in the direction of $\vec{\imath} + \vec{\jmath}$.

$$T = \frac{40}{\sqrt{x^2+y}}$$

$$T_x = -20\,(x^2+y)^{-3/2} \cdot 2x \quad \text{so } T_x(2,5) = \frac{-40(2)}{\sqrt{(4+5)^3}} = \frac{-80}{27}$$

$$T_y = -20\,(x^2+y)^{-3/2} \quad \text{so } T_y(2,5) = \frac{-20}{\sqrt{9^3}} = \frac{-20}{27}$$

$$\text{So } \nabla T(2,5) = \left(\frac{-80}{27}, \frac{-20}{27}\right)$$

$$\text{So } f_{\vec{u}}(2,5) = \left(\frac{-80}{27}, \frac{-20}{27}\right) \cdot \left(\frac{1}{\sqrt{2}}\vec{\imath} + \frac{1}{\sqrt{2}}\vec{\jmath}\right) = \frac{-80}{27\sqrt{2}} - \frac{20}{27\sqrt{2}} = \frac{-100}{27\sqrt{2}}$$

c) In what direction does T decrease the most rapidly?

T decreases most rapidly in direction of $-\nabla T(2,5)$

which here is $\frac{80}{27}\vec{\imath} + \frac{20}{27}\vec{\jmath}$

d) In what direction is the rate of change 0? $\perp$ to $\nabla T(2,5)$,

which is $\frac{80}{27}\vec{\imath} - \frac{20}{27}\vec{\jmath}$

12 points

5. For this problem, we will consider a tree trunk to have the shape of a right circular cylinder. First, right down the volume of the tree trunk as a function of the radius r and the height h.

$$V = \pi r^2 h$$

b) Suppose the diameter of the trunk increases 1 inch per year and the height of the trunk increases 6 inches per year. How fast is the volume of the tree trunk increasing when it is 10 ft high and 14 inches in diameter?

So $\frac{dr}{dt} = 1/2 \qquad \frac{dh}{dt} = 6$

$V < \begin{matrix} r - t \\ h - t \end{matrix}$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$h = 10\text{ft} = 120$

$r = 7 \text{ IN}$

$$= 2\pi r h \left(1/2\right) + \pi r^2 (6)$$

$$= 1\pi(7)(\cancel{10}\ 120) + \pi(7)^2 \cdot 6$$

$$= 840\pi + 294\pi$$

$$= 1134\,\pi.$$

$$\begin{array}{r} 120 \\ \times\ 7 \\ \hline 840 \end{array}$$

$$\begin{array}{r} 840 \\ 294 \\ \hline 1134 \end{array}$$

24 points

(3 points each)

6. Mark the following problems True or False.

A) Let $P = f(m, d)$ be the purchase price in dollars of a used car with m miles on its engine and with original coast d dollars when new. Then $\frac{\partial P}{\partial m}$ and $\frac{\partial P}{\partial d}$ have the same sign.

If increase m, price goes down

If increase d, price goes up

False

B) If $f(x, y)$ is a function with the property that $f_x(x, y)$ and $f_y(x, y)$ are both constant, then f is linear.

This corresponds to saying whenever we walk in the x-direction, we have the same slope. Same for y-direction. By pg 585, that is what happens in a plane.

True

C) If $f(x, y)$ has $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then f is constant everywhere.

Ex $f(x,y) = x^2 + y^2$

has $f_x = 2x$ $\quad f_y = 2y$.

So $f_x(0,0) = 0$ $\quad f_y(0,0) = 0$ but $\not\Rightarrow$ f is constant.

False

D) For every function $f(x, y)$, we have $f_{xy} = f_{yx}$.

False

NO, must have them be cont.

E) If $f(x, y)$ is a function of 2 variables defined for every x and y, then $f(0, y)$ is a function of 1 variable.

True

F) The plane $x + 2y - 3z = 1$ passes through the origin.

$0 + 0 - 0 \neq 1$

False

G) Two contours of $f(x, y)$ with different heights never intersect.

[pg 576]

True

H) The level surfaces of $g(x, y, z) = x + 2y + z$ are parallel planes.

set $g =$ constant, get

True

$c = x + 2y + z$: eqn of plane

are parallel because slopes are same.
(also, normal vectors the same)