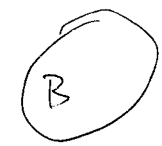
(1) Suppose that the function $f(x, y) = x^2 + 2x + y^2 + 1$ gives the temperature at the point (x, y). Let B be the disk of radius 2 centered at the origin, i.e., $B = \{(x, y) \mid x^2 + y^2 \leq 4\}$.

(a) Find the maximum and minimum values of f restricted to the disk B.



1st, find int. crit. pts. of f. $\nabla f = (2x+2, 2y)$ - always exists, only crit. pt. is $\underline{(-1,0)}$.

2nd, use Lagrange multipliers to find possible extrema on boundary circle $g(x,y) = x^2+y^2 = 4$

$\nabla g = (2x, 2y)$ - never 0 on $g=4$. So solve $\nabla f = \lambda \nabla g$ on $g=4$:

$2x+2 = \lambda 2x$, $\underline{2y = \lambda 2y}$, and $x^2+y^2 = 4$.

$\Downarrow$

$y=0$ or $\lambda = 1$. If $y=0$, then $x = \pm 2$. If $\lambda = 1$, then $2x+2 = 2x$, or $2=0$ - no soln. So solns. are $\underline{(\pm 2, 0)}$.

$f(-1,0) = 1-2+0+1 = 0 \leftarrow$ minimum value

$f(2,0) = 4+4+0+1 = 9 \leftarrow$ maximum value

$f(-2,0) = 4-4+0+1 = 1$

(b) If you start at the point (1, 0) and move northeast at unit speed, how fast is the temperature changing?

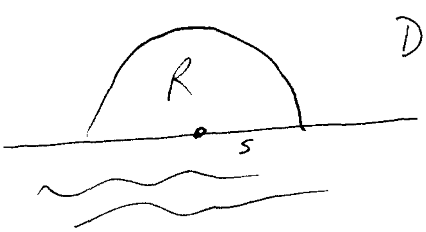
NE $\nearrow$ $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The answer is $f_{\vec{u}} = \nabla f(1,0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= (2\cdot 1+2, 2\cdot 0)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= (4,0)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= \frac{4}{\sqrt{2}} = 2\sqrt{2}$

(2) A city occupies a semicircular region of radius 5 km bordering on the ocean. Find the average distance from points in the city to the ocean.



Distance to ocean is y, so the average distance is

$$\frac{\iint_R y\,dA}{\text{area of } R} = \frac{\iint_R y\,dA}{\frac{1}{2}\pi 5^2} = \frac{\iint_R y\,dA}{\frac{25}{2}\pi}$$

This is easier in polar coordinates:

$$\iint_R y\,dA = \int_0^5\int_0^\pi y\cdot r\,d\theta\,dr = \int_0^5\int_0^\pi r^2\sin\theta\,d\theta\,dr \quad (\text{since } y = r\sin\theta)$$

$$= \int_0^5 r^2\left[-\cos\theta\Big|_0^\pi\right]dr = \int_0^5 r^2\big(-(-1)-(-1)\big)\,dr = \int_0^5 2r^2\,dr = \frac{2}{3}r^3\Big|_0^5 = \frac{2\cdot 125}{3}$$

So the average distance is

$$\frac{\frac{2\cdot125}{3}}{\frac{25}{2}\pi} = \frac{2\cdot2\cdot125}{3\cdot25\cdot\pi} = \frac{20}{3\pi}$$

We can also do it in rectangular coordinates:

$$\iint_R y\,dA = \int_{-5}^5\int_0^{\sqrt{25-x^2}} y\,dy\,dx = \int_{-5}^5\left[\frac{y^2}{2}\Big|_0^{\sqrt{25-x^2}}\right]dx = \int_{-5}^5 \frac{25-x^2}{2}\,dx$$

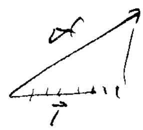
$$= \frac{25x-\frac{1}{3}x^3}{2}\Big|_{-5}^5 = \frac{2\cdot125}{3},$$ as before

(3) State whether each of the following is true or false. If it's false, explain why or give an example showing that it's false.

(a) If $f(x,y) = k$ for all points (x,y) in a region R, then $\iint_R f\,dA = k \cdot \text{Area}(R)$.

True

(b) There is a function f with $||\nabla f|| = 4$ and $f_{\vec{i}} = 5$ at some point.



False – $f_{\vec{i}} = \nabla f \cdot \vec{i}$ = shadow of ∇f on $\vec{i}$.

If ∇f has length 4, its shadow can't be 5.

(c) There is a function f with $||\nabla f|| = 4$ and $f_{\vec{j}} = -3$ at some point.

True

(d) Let $\rho(x,y)$ be the population density of a city, in people per km^2. If R is a region in the city, then $\iint_R f\,dA$ gives the average number of people per km^2.

False – $\dfrac{\iint_R f\,dA}{\iint_R dA = \text{area of } R}$ is the average

(e) Let $f(x,y,z)$ be a continuous function. If W_1 and W_2 are solid regions with volume(W_1) > volume(W_2), then $\iiint_{W_1} f\,dV > \iiint_{W_2} f\,dV$.

False – If $f<0$ everywhere, then $\iiint_{W_1} f\,dV > \iiint_{W_2} f\,dV$

(4) Let $f(x)$ be a smooth function of one variable. Define the function $g(u,v)$ to be $f(uv^2)$. Find g_{uv} and g_{vu} and show that they are equal.

$g_{uv} = f_{uv}$

$g_{vu} = f_{vu}$

$x = uv^2$

$$\begin{tikzcd} & f \arrow[d, "\frac{df}{dx}"] & \\ & x \arrow[dl, "\frac{\partial x}{\partial u} = v^2"'] \arrow[dr, "\frac{\partial x}{\partial v} = 2uv"] & \\ u & & v \end{tikzcd}$$

$$f_u = \frac{df}{dx} \cdot \frac{\partial x}{\partial u} = \frac{df}{dx} \cdot v^2$$

So $f_{uv} = (f_u)_v = \left(\frac{df}{dx} \cdot v^2\right)_v = \left(\frac{df}{dx}\right)_v \cdot v^2 + \frac{df}{dx} \cdot 2v$

$$= \frac{d^2f}{dx^2} \cdot 2uv \cdot v^2 + \frac{df}{dx} \cdot 2v$$

$$= \frac{d^2f}{dx^2} \cdot 2uv^3 + \frac{df}{dx} \cdot 2v$$

$$\begin{tikzcd} & \frac{df}{dx} \arrow[d, "\frac{d^2f}{dx^2}"] & \\ & x \arrow[dl, "v^2"'] \arrow[dr, "2uv"] & \\ u & & v \end{tikzcd}$$

Similarly, $f_v = \frac{df}{dx} \cdot \frac{\partial x}{\partial v} = \frac{df}{dx} \cdot 2uv$

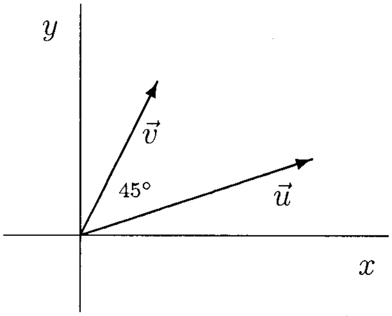
and $(f_{vu}) = (f_v)_u = \left(\frac{df}{dx}\right)_u \cdot 2uv + \frac{df}{dx} \cdot 2v$

$$= \frac{d^2f}{dx^2} \cdot v^2 \cdot 2uv + \frac{df}{dx} \cdot 2v$$

$$= \frac{d^2f}{dx^2} \cdot 2uv^3 + \frac{df}{dx} \cdot 2v$$

They are equal.

(5) Let $\vec{u}$ and $\vec{v}$ be two 3-dimensional vectors lying in the xy-plane, with an angle of 45° between them. If $||\vec{u}|| = 3$ and $||\vec{v}|| = 2$, find the following:



(a) $\vec{u} \cdot \vec{v}$ $= ||\vec{u}|| \cdot ||\vec{v}|| \cos 45° = 3 \cdot 2 \cdot \frac{1}{\sqrt{2}} = 3\sqrt{2}$

(b) $\vec{u} \times \vec{v}$ length: $||\vec{u}|| ||\vec{v}|| \sin 45° = 3 \cdot 2 \cdot \frac{1}{\sqrt{2}} = 3\sqrt{2}$

direction: by right hand rule, $\vec{k}$

So $\vec{u} \times \vec{v} = 3\sqrt{2}\,\vec{k} = (0, 0, 3\sqrt{2})$

(6) The mass (in kg) of a melting block of ice is given by $S(x, t)$, where x is the initial mass of the block and t is the number of minutes since it was taken out of the freezer.

(a) What are the units of $\frac{\partial S}{\partial x}$? Do you expect it to be positive or negative? Why?

Units – kg/kg. Should be positive – the bigger it was when it started, the bigger it is now.

(b) What are the units of $\frac{\partial S}{\partial t}$? Do you expect it to be positive or negative? Why?

Units – kg/minute. Should be negative – the longer it's been melting, the smaller it is.

(7) Wilma Whale is swimming in the ocean. The density of plankton, her favorite food, in grams per cubic meter, at the point (x, y, z) in the ocean, is given by a differentiable function $f(x, y, z)$. From Wilma's present location, the plankton density decreases most rapidly in the direction of the vector $(2, 3, 6)$. The rate of change in that direction is -14 grams per meter^4.

(a) At what rate is the density changing if she moves straight up (i.e., in the $\vec{k}$ direction) at 3 meters per second?

Answer: $3(\nabla f \cdot \vec{k})$. ∇f is a vector in the direction $(-2, -3, -6)$ with magnitude 14 (it points opposite the direction of greatest decrease),

so $\nabla f = \dfrac{(-2,-3,-6)}{\|(-2,-3,-6)\|} \cdot 14 = \dfrac{(-2,-3,-6)}{\sqrt{4+9+36}} \cdot 14 = \dfrac{(-2,-3,-6)}{\sqrt{49}} \cdot 14 = \dfrac{(-2,-3,-6)}{7} \cdot 14$

$= (-4, -6, -12)$.

So $3 \cdot f_{\vec{k}} = 3 \cdot ((-4,-6,-12) \cdot (0,0,1)) = -36$ grams/meter^3/second

(b) Now assume that Wilma likes the plankton density just the way it is at her current location (any lower and she'd get hungry, any higher and it'd stick in her teeth). What's a direction that she could move in without changing plankton density? (There are lots of possible answers, but "she should sit still" doesn't count.)

She should move in a direction $\perp$ to $\nabla f = (-4, -6, -12)$,

for example, in the direction of $(6, -4, 0)$.

(Any answer $\vec{v}$ with $\vec{v} \cdot (-4, -6, -12) = 0$ works.)

(8) I've decided to open a Krunchy Kream donut franchise in Springfield. I've determined that my profits (in millions of dollars per month) will be given by the function $f(x, y) = \dfrac{e^{x-1}}{2y}$, where x is a measure of the deliciousness of my donuts and y is a measure of the deliciousness of my competitors' donuts.

(a) Find an equation for the tangent plane to the graph of f at the point $(1, 2, 1/4)$.

$$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$f_x = \frac{e^{x-1}}{2y}, \text{ so } f_x(1,2) = \frac{1}{4} \qquad f_y = \frac{-e^{x-1}}{2y^2}, \text{ so } f_y(1,2) = \frac{-1}{8}$$

So the plane is $z = \frac{1}{4} + \frac{1}{4}(x-1) - \frac{1}{8}(y-2)$

or $z = \frac{1}{4}x - \frac{1}{8}y + \frac{1}{4}$

(b) Use linear approximation to estimate my profits if $x = 1.2$ and $y = 1.9$.

Use the tangent plane: $f(1.2, 1.9) \approx \frac{1}{4} + \frac{1}{4}(1.2-1) - \frac{1}{8}(1.9-2)$

$= \frac{1}{4} + \frac{1}{4}(.2) - \frac{1}{8}(-0.1)$

EXTRA CREDIT (5 pts) Draw the monkey in his saddle.