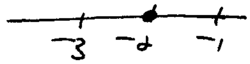
Math 6B Practice Final Solns

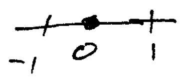
1. a) Use $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2} = 1$. So the radius of conv. is 1. Check endpts:



If $x=-1$, series is $\sum \frac{1}{n^2}$ - convergent p-series.

If $x=-3$, series is $\sum \frac{(-1)^n}{n^2}$ - convergent alt. series. So the interval of convergence is $[-3,-1]$

b) Again, use $R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$. So $R=1$; check endpts



If $x=1$, series is $\sum n$ - divergent by n^{th} term test

If $x=-1$, series is $\sum (-1)^n n$ - div. by n^{th}-term test

So the interval of convergence is $(-1,1)$.

c) Since the series is $(x-7)^2/4 + (x-7)^4/16 + \cdots$, c_m is 0 for m odd, so $R = \lim_{n\to\infty}\left|\frac{c_m}{c_{m+1}}\right|$ doesn't exist. Apply the Ratio Test to the power series!

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-7)^{2n+2}}{4^{n+1}} \bigg/ \frac{(x-7)^{2n}}{4^n}\right| = \lim_{n\to\infty} \frac{(x-7)^2}{4}.$$

The series converges if this limit is <1, ie, $(x-7)^2 < 4$, ie, $|x-7|<2$, and diverges if " " " >1, ie, $(x-7)^2 > 4$, ie, $|x-7|>2$. So the radius of conv. is 2. Check the endpts:

If $x=5$, the series is $\sum \frac{(-2)^{2n}}{4^n} = \sum 1$, diverges.

If $x=9$, " " " $\sum \frac{(2)^{2n}}{4^n} = \sum 1$, diverges

So the interval of convergence is $(5,9)$.

(2) a) Converges by the root test.

b) Converges by the ratio test.

c) $\sum \ln\left(\frac{n}{2n+1}\right)$ diverges by n^{th}-term test

d) Converges by limit comparison with p-series $\sum \frac{1}{n^{3/2}}$

e) Converges - Throw out the first 3 terms and you have a decreasing alternating series.

(3) The series converges by the alt. series test. We know that $|S - S_n| \le a_{n+1}$, so we need $\frac{1}{(n+1)^2} < \frac{1}{10{,}000}$, or $(n+1)^2 > 10{,}000$, or $n+1 > 100$, or $n = 100$ terms.

(4) $y(x) = c_0 + c_1 x + c_2 x^2 + \cdots$. Since $y(0) = 2 = c_0 + c_1 \cdot 0 + c_2 \cdot 0 + \cdots c_0$,

$y(x) = 2 + c_1 x + c_2 x^2 + \cdots$

Now, $y' + y = 1$

$c_1 + 2c_2 x + 3c_3 x^2 + \cdots + 2 + c_1 x + c_2 x^2 + \cdots = 1$

So: $c_1 + 2 = 1 \Rightarrow c_1 = -1$; $2c_2 + c_1 = 0 \Rightarrow c_2 = \frac{1}{2}$; $3c_3 + c_2 = 0 \Rightarrow c_3 = \frac{-1}{3 \cdot 2}$

$\cdots$ Get $y = 2 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$

(5) One soln: $\frac{dP}{dt} = P(P - L)$, where L is the minimum # of rhinos necessary for them to find each other in order to mate.

(6) a) $P' = 0$ when $P = 0, 50,$ or 200

b) $P' > 0$ for $50 < P < 200$

c) $P' < 0$ for $0 < 50 < P$ & $P > 200$.

d) If you start w/ 150, in the long run they'll increase to 200.

(7) a) False: Ex: $b_n = (-1)^n \cdot \frac{1}{n}$. Then $\sum (-1)^n b_n = \sum (-1)^n (-1)^n \cdot \frac{1}{n} = \sum \frac{1}{n}$, which diverges.

Note: this wouldn't be true even if we assumed that $b_n \geq 0$.

Ex: $\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \cdots$

b) True, by the integral test (assuming that $f(x)$ is continuous for $x \geq 1$).

c) False. Ex: $a_n = 0$ for all n, $b_n = 1$ for all n.

(8)

$$1000 \cdot (1.05) + 1000 \cdot (1.05)^2 + \cdots + 1000 \cdot (1.05)^{30}$$
$$= 1000 \cdot 1.05 \left(1 + 1.05 + \cdots + (1.05)^{29}\right)$$
$$= 1050 \left(\frac{1 - (1.05)^{30}}{1 - 1.05} \right) \approx \$69,760.80$$

(9) a) Hard way – compute derivatives. Easy way: $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ for $|x| < 1$,

so $\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + \cdots$ for $|x^2| < 1$, i.e., $|x| < 1$.

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

b) $\int \frac{1}{1+x^2}\,dx = \arctan$, & $\arctan(0) = 0$, so

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$ with radius of conv. still $= 1$.

Check endpts: If $x = 1$, get $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ – converges by alt. series test

If $x = -1$, get $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$: converges by alt. series test ~~diverges by comparison to $\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots)$~~.

So the interval of convergence is $[-1, 1]$.

c) ~~$E_{21}(\frac{1}{\sqrt{3}})/3$~~ The 21^{st}-degree Taylor poly. is also the 22^{nd}-dg. Taylor polynomial, since the coeff. of x^{22} in the Taylor series is 0. So ~~we want $E_{22}(\frac{1}{\sqrt{3}})$, and we know that $|E_{22}(\frac{1}{\sqrt{3}})| \leq \frac{M \cdot (\frac{1}{\sqrt{3}})^{23}}{23!}$,~~

~~where M is a bound on $|\arctan^{(23)}(x)|$ on $[0, \frac{1}{\sqrt{3}}]$.~~ This is an alternating series, so the error is $\leq \frac{\left(\frac{1}{\sqrt{3}}\right)^{23}}{23}$.

(10) a) $1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$

b) $|E_4(0.1)| \le \frac{M(0.1)^5}{5!}$, where M is a bound for $(e^{2x})^{(5)}$ on $[0, 0.1]$. Since $(e^{2x})^{(5)} = 32e^{2x}$ is increasing, it is $\le 32e^{0.1}$ on $[0,1]$. Since I don't know $e^{0.1}$, but I do know that $e^{0.1} < e < 3$, I'll take $M = 3$, and approximate the error by $\frac{3(0.1)^5}{5!}$

(11) a) $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \cdots$

$|E_n(1)| \le \frac{M\, 1^{n+1}}{(n+1)!} = \frac{M}{(n+1)!}$, where M is a bound for $\sin^{(n)} x$. Each derivative is $\pm \sin$ or $\pm \cos$, so I can take $M = 1$. So I need

$\frac{1}{(n+1)!} < .001$, or $(n+1)! > 1000$. $5! = 5 \cdot 4 \cdot 3 \cdot 2 = 120$, so $6! = 720$, so $7! > 1000$. So $\sin 1 = 1 - 1/3! + 1/5! - 1/7! \pm .001$.

b) $\ln x = \ln 1 + \ln'(1)(x-1) + \ln''(1)(x-1)^2/2! + \cdots$

$\ln x = 0 + (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 + \cdots$

$|E_n(1.1)| < \frac{M(0.1)^{n+1}}{(n+1)!}$, where M is a bound for $\ln^{(n+1)}(x)$ on $[1, 1.1]$.

Since $|\ln^{(n+1)}(x)| = \frac{n!}{x^{n+1}}$, I can use $M = n!$

So I need $\frac{n!(0.1)^{n+1}}{(n+1)!} < .001$, or $\frac{(0.1)^{n+1}}{n+1} < .001$, or $\frac{n+1}{(0.1)^{n+1}} > 1000$

$n = 2$ works, so $\ln 1.1 = 0 + (1.1-1) + (1.1-1)^2/2 \pm .001$

(12) $|E_3(2)| \le \frac{M \cdot 2^4}{4!}$, where M is a bound on $|f^{4}(x)| \le \frac{1}{3+5x^2}$ on $[0,2]$.

This is biggest when $x=0$, so I can use $M = \frac{1}{3}$. So the error is $\le \frac{3 \cdot 2^4}{4!} = \frac{3 \cdot 16}{24} = 2$.

(13) a) $P_3(x) = \cancel{1 - x + x^2}\ 1 - (x-1) + (x-1)^2 - (x-1)^3$

A bound for $\left|\left(\frac{1}{x}\right)^{(4)}\right| = \frac{24}{x^5}$ on $[.8, 1.2]$ is $M = \frac{24}{(.8)^5}$,
so $|E_3(x)| \le \left|\frac{\frac{24}{(.8)^5}\cdot (x-1)^4}{4!}\right| = \left|\frac{(x-1)^4}{(.8)^5}\right|$ on $[.8, 1.2]$.

b) $P_2(x) = 1 + x^2$. A bound for $\left|\left(e^{x^2}\right)^{(3)}\right| = \left|12xe^{x^2} + 8x^3e^{x^2}\right|$ on $[-0.1, 0.1]$ is $M = 12 \cdot .01\, e^{.0001} + 8 \cdot .000001 \cdot e^{.0001}$, so

$|E_2(x)| \le \frac{M \cdot |x^3|}{3!}$ on $[-0.1, 0.1]$. (Can substitute 3 for $e^{.0001}$ to get a definite estimate.)

c) $P_3(x) = 8 + 3/8\,(x-16) - \frac{3}{16 \cdot 32 \cdot 2!}(x-16)^2 + \frac{15}{64 \cdot 2^9 \cdot 3!}(x-16)^3$
A bound for $\left|\left(x^{3/4}\right)^{(4)}\right| = \left|\frac{135}{128 \cdot 2^{13}} \cdot \frac{1}{x^{13/4}}\right|$ on $[15, 17]$ is

$M = \frac{135}{128 \cdot 2^{13}} \cdot \frac{1}{15^{13/4}}$ (or use $M = \frac{135}{128 \cdot 2^{13}} \cdot \frac{1}{15^3}$),
so $|E_3(x)| \le \frac{M \cdot |x-16|^4}{4!}$ on $[15, 17]$.

(14) Use $f(x) = x^{1/2}$ at $a = 1$.
$P_1(x) = 1 + \frac{1}{2}(x-1)$.

A bound for $|f''(x)| = \frac{1}{4}\frac{1}{x^{3/2}}$ on $[1, 1.05]$ is $M = 1/4$, so
$E_1(1.05) \le \frac{1/4 \cdot (.05)^2}{2!} = \frac{.0025}{8}$,

so $\sqrt{1.05} = 1 + \frac{1}{2}(.05) \pm \frac{.0025}{8}$
$= 1.025 \pm \frac{.0025}{8}$.