(1) (20 pts) Determine whether the following series converge or diverge. Be sure to explain your reasoning.

(a) $\sum_{n=0}^{\infty} \frac{n}{47n^2-217}$ limit comparison with $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{\frac{n}{47n^2-217}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{47n^2-217} = \frac{1}{47}$$

Since $\sum \frac{1}{n}$ (harmonic series) diverges, so does our series

(b) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$ n^{th} term test:

$\left(\frac{n+1}{n}\right)^n > 1^n = 1$, so the nth term does not go to 0, so the series diverges

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n^2-1}$

$$= 1 - \frac{2}{3} + \frac{3}{17} - \frac{4}{31} + \cdots$$

This is a decreasing alternating series, and the n^{th} term goes to 0 $\left(\lim_{n\to\infty} \frac{(-1)^{n+1}n}{2n^2-1} = \lim_{n\to\infty} \frac{n}{n} \cdot \frac{(-1)^{n+1}}{2n-\frac{1}{n}} = 0\right)$, so it converges, by the alternating series test.

(2) (20 pts) Find the third-degree Taylor polynomial at the point $a = 1$ for the function

$$f(x) = \sqrt{x}.$$

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$$f(x) = x^{\frac{1}{2}}, \quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = \frac{-1}{4}x^{-3/2}, \quad f'''(x) = \frac{3}{8}x^{-5/2}$$

So $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = \frac{-1}{4}$, $f'''(1) = 3/8$.

So $P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!} + 3/8 \, \frac{(x-1)^3}{3!}$

$$= 1 + \frac{1}{2}(x-1) - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}$$

(3) (20 pts) Compute $1/e$ to within $1/10$ of its actual value. Be sure to explain how you can be sure of the accuracy of your estimate.
(HINT: $1/e = e^{-1}$, and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$)

$$e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \dots$$

$$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \dots$$

This is a decreasing alternating series, so the error

$|e^{-1} - S_n| \le a_{n+1}$, where S_n is the nth partial sum & a_{n+1} is the first term of the series that we didn't add.

To make $\frac{1}{n!} \le \frac{1}{10}$, we need $n = 4$ $\left(\frac{1}{3!} = \frac{1}{6} \text{ \& } \frac{1}{4!} = \frac{1}{24}\right)$.

So, $1 - 1 + \frac{1}{2} - \frac{1}{6}$ is within $\frac{1}{24}$ of e^{-1}, ie,

$$e^{-1} \approx 1/3.$$

(4) (20 pts) Find the interval and radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$. (Be sure to check the endpoints.)

Use the ratio test:

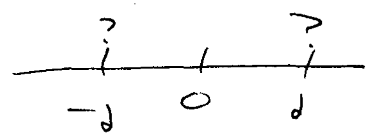
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{x^{2n+2}}{4^{n+1}}}{\frac{x^{2n}}{4^n}} = \lim_{n\to\infty} \frac{x^{2n+2}}{4^{n+1}} \cdot \frac{4^n}{x^{2n}}$$

$$= \lim_{n\to\infty} \frac{x^2}{4} = \frac{x^2}{4}$$

So the series converges absolutely if $\frac{x^2}{4} < 1$, ie, $x^2 < 4$, ie, $-2 < x < 2$,

& diverges if $\frac{x^2}{4} > 1$, ie, $x < -2$ or $x > 2$.

So the radius of convergence is 2.



Check the endpoints:

$x=2$: $\sum \frac{2^{2n}}{4^n} = \sum \frac{4^n}{4^n} = \sum 1$ – diverges

$x=-2$: $\sum \frac{(-2)^{2n}}{4^n} = \sum \frac{4^n}{4^n} = \sum 1$ – diverges

So the interval of convergence is $-2 < x < 2$, ie, $(-2, 2)$.

(5) (20 pts) Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false.

(a) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{3}{2}$. FALSE

$1 + \frac{1}{3} + \frac{1}{9} + \dots = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = 3/2$

So $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} = 3/2 - 1 = 1/2$

(b) If $\sum |a_n|$ converges, then $\sum (-1)^n |a_n|$ converges.

TRUE - Absolute convergence implies convergence

(c) If the terms, a_n, of a series tend to zero as n increases, then the series $\sum a_n$ converges.

$\boxed{\text{FALSE}}$ The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

(d) If the power series $\sum_{n=0}^{\infty} c_n (x-3)^n$ converges at $x = 5$, then it must converge at $x = 1$ as well.

FALSE. 5 could be an endpoint of the interval of convergence, which would make $5-3 = 2$ the ~~other end~~ radius of convergence, which would make $3-2 = 1$ the other endpoint, and we don't know what happens there.