(1) (15 pts)

(a) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{n}}$ converge or diverge? Explain why.

Diverges, by limit comparison to $\sum \frac{1}{n}$

(b) Why does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\frac{1}{n}}$ converge? How close is the second partial sum to the actual s
of the series? (I want an actual number.)

Converges by alt. series test.

$$|S-S_2| \leq |a_3| = \left| \frac{(-1)^3}{3+\frac{1}{3}} \right| = \frac{1}{\frac{10}{3}} = \boxed{3/10}$$

(2) (20 pts) My butler Buttlesworth is retiring this year. To reward him for years of faithful service, I plan to give him \$100,000 each May 9, starting today, until he dies. I want to put all the necessary money in a bank account (earning 10% interest, compounded annually) right now.

(We are supposing that every May 8, I receive the interest for the previous year (exactly 0.10 times the amount in the bank). Thus, if I deposit \$100 today and don't spend any of it over the next year, then on May 8, 2004, I will have \$100(1.10).)

(a) If I think that Buttlesworth is going to live another $19\frac{1}{2}$ years (meaning that he gets 20 payments of \$100,000 – one today, one in one year, ..., and one in nineteen years), how much money do I need to deposit in the bank account right now to cover his payments? (If I thought he'd die in 6 months, I'd need to deposit \$100,000 right now to cover today's payment.)

$\frac{1}{2}$ year : need to deposit \$100,000

need to deposit $\$100{,}000\left(\frac{1}{1.1}\right)$ now in order to have \$100,000 in a year

$1\frac{1}{2}$ years : need $\$100{,}000 + \$100{,}000\left(\frac{1}{1.1}\right)$

$2\frac{1}{2}$ years : need $\$100{,}000 + \$100{,}000\left(\frac{1}{1.1}\right) + \$100{,}000\left(\frac{1}{1.1}\right)^2$

$\vdots$

$19\frac{1}{2}$ years : need $\$100{,}000 + \$100{,}000\left(\frac{1}{1.1}\right) + \dots + \$100{,}000\left(\frac{1}{1.1}\right)^{19}$

$$= \$100{,}000\left(1 + \left(\frac{1}{1.1}\right) + \dots \left(\frac{1}{1.1}\right)^{19}\right)$$

$$= \$100{,}000 \cdot \left(\frac{1-\left(\frac{1}{1.1}\right)^{20}}{1-\frac{1}{1.1}}\right) \simeq \$936{,}492.01$$

(b) If I think that Buttlesworth is actually a robot who will live forever, how much money do I need to deposit in the bank account right now to cover his payments?

$$\$100{,}000 + \$100{,}000 \cdot \left(\frac{1}{1.1}\right) + \$100{,}000\left(\frac{1}{1.1}\right)^2 + \dots$$

$$= \$100{,}000\left(1 + \frac{1}{1.1} + \left(\frac{1}{1.1}\right)^2 + \dots\right)$$

$$= \$100{,}000\left(1\Big/1-\frac{1}{1.1}\right) = \$1{,}100{,}000$$

(3) (20 pts) Out of the kindness of Prof. Klotz' and Prof. Wiseman's hearts, here are the derivatives of $f(x) = \ln x$:

$$f'(x) = \frac{1}{x}, f''(x) = \frac{-1}{x^2}, f'''(x) = \frac{2!}{x^3}, f^{(4)}(x) = \frac{-3!}{x^4}, \ldots, f^{(n)}(x) = \frac{(-1)^{(n-1)}(n-1)!}{x^n}, \ldots$$

(a) What is the Taylor series with base point $a = 2$ for $\ln x$?

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$\ln 2 + \frac{1}{2}(x-2) - \frac{1}{2^2 2!}(x-2)^2 + \frac{2!}{2^3 3!}(x-2)^3 - \frac{3!}{2^4 4!}(x-2)^4 + \cdots$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{2^2 \cdot 2}(x-2)^2 + \frac{1}{2^3 \cdot 3}(x-2)^3 - \frac{1}{2^4 4}(x-2)^4 + \cdots + \frac{(-1)^{n-1}}{2^n n}(x-2)^n + \cdots$$

(b) What is the radius of convergence?

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim \frac{\frac{1}{2^n n}}{\frac{1}{2^{n+1}(n+1)}} = \lim \frac{2^{n+1} \cdot (n+1)}{2^n \cdot n} = 2 \cdot \lim_{n\to\infty} \frac{n+1}{n} = 2 \cdot 1 = 2$$

(c) What is the interval of convergence?

Check endpts. $x = 0$ & $x = 4$

$\underline{x=0}$ Get $\ln 2 - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} \ldots$ diverges (after 1st term, it's harmonic)

$\underline{x=4}$ Get $\ln 2 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$ converges (alt. series test)

So, interval of convergence is $(0, 4]$

(Continued on next page)

(3) (continued)

(d) If you use the third-degree Taylor polynomial to approximate $\ln x$ for $1 < x < 3$, what's the most you could possibly be off by?

$|\ln x - P_3(x)| = E_3(x) \leq \left|\frac{M(x-2)^4}{4!}\right|$, where M is a bound for $|\ln^{(4)}(x)|$ on $[1,3]$.

$|\ln^{(4)}(x)| = \left|\frac{-3!}{x^4}\right| = \frac{3!}{x^4}$, which is largest when x is smallest, so when $x=1$. Get $\frac{3!}{1} = 6$ for M.

So $E_3(x) \leq \frac{6(x-2)^4}{4!}$ on $[1,3]$. This is largest when $x=1$ or 3;

Then it's $\frac{6 \cdot (1)^4}{4!} = \frac{6}{4!} = \frac{1}{12}$.

(4) (10 pts) Using only the definition of convergence of series, determine whether the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}-\frac{n+2}{n+1}\right)$ converges or diverges. If it converges, find the sum.

Converges if & only if sequence of partial sums converges.

Series: $\left(\frac{2}{1}-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{4}{3}\right)+\left(\frac{4}{3}-\frac{5}{4}\right)+\cdots$

$S_1 = 2-\frac{3}{2}$; $S_2 = 2-\frac{3}{2}+\frac{3}{2}-\frac{4}{3} = 2-\frac{4}{3}$; $S_3 = S_2+\left(\frac{4}{3}-\frac{5}{4}\right) = 2-5/4$

$S_n = 2-\frac{n+2}{n+1}$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2-\frac{n+2}{n+1}\right) = \lim_{n\to\infty}(2-1) = 1$

(5) (10 pts) Find the first three nonzero terms of the power series solution of the form $\sum c_n x^n$ for the differential equation $y' = xy$ with initial condition $y(0) = 1$.

$y = C_0 + C_1 x + C_2 x^2 + \cdots$

$y(0) = 1 = C_0 + C_1\cdot 0 + C_2\cdot 0^2 + \cdots = C_0$, so $C_0 = 1$

$y' = xy$, so

$C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots = x + C_1 x^2 + C_2 x^3 + C_3 x^4 + \cdots$

So: $C_1 = 0$

$2C_2 = 1 \Rightarrow C_2 = \frac{1}{2}$

$3C_3 = C_1 = 0 \Rightarrow C_3 = 0$

$4C_4 = C_2 = \frac{1}{2} \Rightarrow C_4 = \frac{1}{8}$

So $y = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \cdots$

(6) (15 pts) Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false. (Each a_n is a (not necessarily positive) real number.)

(a) If the terms, a_n, of a series tend to zero as n increases, then the series $\sum a_n$ converges.

$\boxed{\text{FALSE!!!}}$ Ex: harmonic series $1+\frac{1}{2}+\frac{1}{3}+\ldots$

(b) If $\sum a_n x^n$ diverges when $x = 1$, then it diverges when $x = -2$.

True (radius of conv. is at most 1)

(c) If $\sum a_n x^n$ converges when $x = 7$, then it converges when $x = -9$.

False - radius of convergence could be between 7 & 9.

(d) If $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

False - remember, a_n isn't nec. positive.

Ex: $\underset{a_0}{1} - \underset{a_1}{\frac{1}{2}} + \underset{a_2}{\frac{1}{3}} - \underset{a_3}{\frac{1}{4}}$ converges, but $\underset{(-1)^0 a_0}{1} + \underset{(-1)^1 a_1}{\frac{1}{2}} + \underset{(-1)^2 a_2}{\frac{1}{3}} + \underset{(-1)^3 a_3}{\frac{1}{4}}$ diverges

(e) Let $s_1, s_2, \ldots$ be a *sequence* with $\lim_{n\to\infty} s_n = L$. If $s_{10} = s_{20} = s_{30} = \ldots = 12$, then $L = 12$.

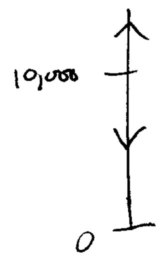
True - otherwise you'd have have infinitely many pts. away from the limit, which is impossible.

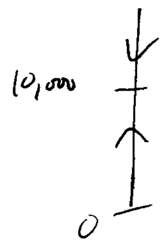
(7) (10 pts) We want to study the populations of whales in the ocean and of mold on the bread in my refrigerator. We know that whales live in the vast, vast ocean and have plenty of room and lots of delicious plankton to eat. At the same time, though, they're so spread out that if there aren't very many of them, they won't be able to find each other in order to breed.

We also know that mold reproduces asexually (i.e., a mold cell doesn't need another mold cell to reproduce). On the other hand, once the mold covers the bread, there'll be no place else for it to live.

Differential equations modeling the two populations $P(t)$ and $Q(t)$ are given below. Which is which? That is, is $P(t)$ the whale population at time t and $Q(t)$ the mold population, or is it the other way around? Explain your reasoning. (k_1 and k_2 are positive constants.)

$$\frac{dP}{dt} = k_1 P(P - 10,000) \qquad \frac{dQ}{dt} = k_2 Q(10,000 - Q)$$





Whales

die if pop. is small,
grow if pop. is big

Mold

grow if pop. is small
die if pop. is big

XTRA CREDIT (5 pts) Explain why it's important for us to be able to estimate how well the th partial sum of a convergent Taylor series approximates the true sum of the series.

It's useless to approximate a function if you don't know how good your approximation is – "The patient needs around 100 mg of medicine" isn't going to do it, Dr. Fine.