1. (10 points) Find an equation for the tangent line to the curve $y = 3(x+1)^2$ at the point (-2, 3).

$$\text{slope} = \left.\frac{dy}{dx}\right|_{x=-2} \qquad \frac{dy}{dx} = 3\cdot 2(x+1) = 6x+6, \text{ so } \left.\frac{dy}{dx}\right|_{x=-2} = -12+6 = -6$$

line: $y - y_0 = -6(x - x_0)$ $\quad (x_0, y_0) = (-2, 3)$, so

$$y - (3) = -6(x-(-2))$$
$$y - 3 = -6x - 12$$
$$y = -6x - 9$$

2. (10 points) Newton's law of gravitation states that the gravitational force F between two bodies of mass m_1 and m_2, respectively, is given by the equation $F(r) = G\frac{m_1 m_2}{r^2}$, where r is the distance between the centers of mass of the bodies and G is the gravitational constant. Compute $\frac{dF}{dr}$, and explain the physical significance of the sign of the derivative.

$$\frac{dF}{dr} = \frac{d}{dr}\left(G\frac{m_1 m_2}{r^2}\right) = \frac{d}{dr}\left(Gm_1m_2 \cdot \frac{1}{r^2}\right)$$
$$= \frac{d}{dr}\left(Gm_1m_2 \cdot r^{-2}\right)$$
$$= Gm_1m_2 \cdot -2r^{-3}$$
$$= \frac{-2Gm_1m_2}{r^3}$$

This is negative, meaning that the gravitational force between two objects decreases as they move farther apart.

3. (12 points) Let $f(x)=e^{2x}$.

a) Is $f(x)$ increasing or decreasing at x=1? How do you know?

$f'(x) = 2e^{2x}$, so $f'(1) = 2e^2 > 0$, so $f(x)$ is <u>increasing</u> at x

b) Is $f'(x)$ increasing or decreasing at x=1? How do you know?

$f''(x) = (f'(x))' = (2e^{2x})' = 4e^{2x}$, so $f''(1) = 4e^2 > 0$, so $f'(x)$ is <u>increasing</u> at $x=1$.

c) Let n be a positive integer (e.g., 1, 2, or 3). What is $\frac{d^n f}{dx^n}$?

$f'(x) = 2e^{2x}$, $f''(x) = 2 \cdot 2e^{2x} = 2^2 e^{2x}$, $f'''(x) = 2 \cdot 2^2 e^{2x} = 2^3 e^{2x}$, ...

$$\frac{d^n f}{dx^n} = 2^n e^{2x}$$

4. (12 points) USE THE DEFINITION OF DERIVATIVE to compute the derivative of the function $f(x)=x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} 2x+h \quad (\text{since } h \neq 0)$$

$$= 2x$$

5. (12 points) Compute the derivatives of the following functions.

a) $f(x)=\dfrac{\sin x}{x+1}$ $\qquad \dfrac{u}{v} \qquad \dfrac{vu'-uv'}{v^2} = \dfrac{(x+1)\cos x - \sin x \cdot 1}{(x+1)^2}$

$$= \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

b) $f(x)=(3x^{21}-x^{-7.2})\overset{(4^x)}{\cancel{(\sqrt{3x^2+8x+10})}}$

$$\cancel{= (3x^{21}-x^{-7.2})'\left((3x^2+8x+10)^{\frac{1}{2}}\right)' + (3x^{21}-x^{-7.2})'\left((3x^2+8x+10)^{\frac{1}{2}}\right)}$$

$$\cancel{= (3x^{21}-x^{-7.2})\cdot\tfrac{1}{2}(3x^2+8x+10)^{-\frac{1}{2}}\cdot(6x+8) + (3\cdot 21\cdot x^{20}+7.2x^{-8.2})\left((3x^2+8x+10)^{\frac{1}{2}}\right)}$$

$$= (3x^{21}-x^{-7.2})'\cdot 4^x + (3x^{21}-x^{-7.2})\cdot(4^x)' = (63x^{20}+7.2x^{-8.2})\cdot 4^x + (3x^{21}-x^{-7.2})\cdot \ln 4\cdot 4^x$$

c) $f(x)=\ln(\cos(x^2))$ $\qquad \dfrac{1}{\cos(x^2)}\cdot(\cos(x^2))' = \dfrac{1}{\cos(x^2)}\cdot(-\sin(x^2))\cdot 2x$

$$= \frac{-2x\sin(x^2)}{\cos(x^2)}$$

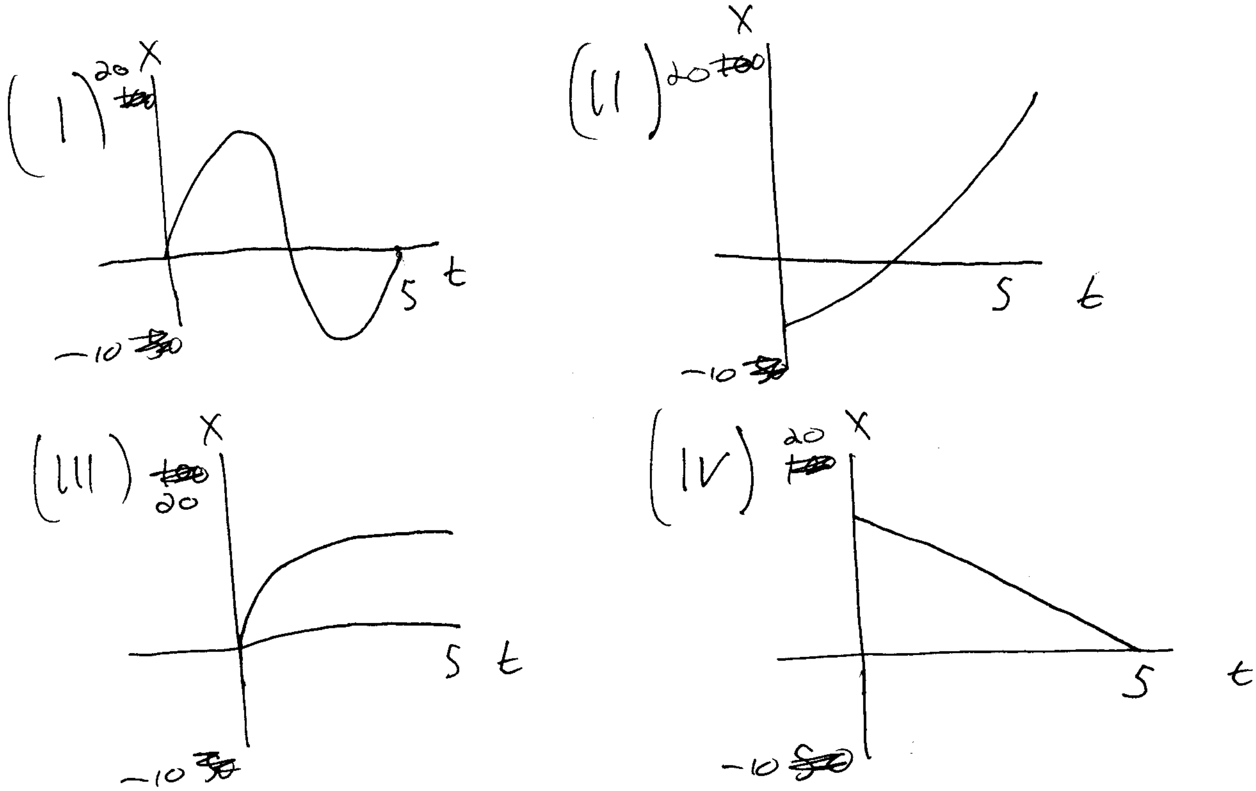
$$= -2x\tan(x^2)$$

d) $f(x)=x^x$

$$(x^x)' = \left((e^{\ln x})^x\right)' = \left(e^{x\ln x}\right)' = e^{x\ln x}\cdot(x\ln x)'$$

$$= e^{x\ln x}\cdot\left(x\cdot\tfrac{1}{x} + 1\cdot\ln x\right)$$

$$= e^{x\ln x}\cdot(1+\ln x)$$



6. (12 points) Each of the graphs above shows the position of a particle moving along the x-axis as a function of time, $0 \leq t \leq 5$. The vertical scales of the graphs are the same.<<< During this time interval, which particle has

a) Constant velocity? IV – constant slope

b) The greatest initial velocity? III – biggest slope at $t = 0$

c) The greatest average velocity? II – biggest change in x from $t=0$ to $t=5$

d) Zero average velocity? I – no change in x between $t=0$ & $t=5$

e) Zero acceleration? IV – same as constant velocity

f) Positive acceleration throughout? II – slope is increasing throughout

7. (10 points) For each of the following statements, explain whether it's true or false. If it's false, give an example showing that it's false.

a) If $f'(x_0) = 0$, then $f''(x_0) = 0$ as well.

False. Take $f(x) = x^2$, $x_0 = 0$. Then $f'(x_0) = 2x_0 = 0$, but
$f''(x_0) = 2 \neq 0$.

b) If f and g are differentiable at x_0, then $\frac{f}{g}$ is differentiable at x_0 as well.

False - we also need $g(x_0) \neq 0$. For example, take $f(x) = x$, $g(x) = x^2$, and $x_0 = 0$. Then f & g are differentiable at $x_0 = 0$, but $\frac{f}{g}(x) = \frac{x}{x^2} = \frac{1}{x}$ is not even defined, much less differentiable, at $x_0 = 0$.

c) If f is differentiable, then it's continuous.

True.

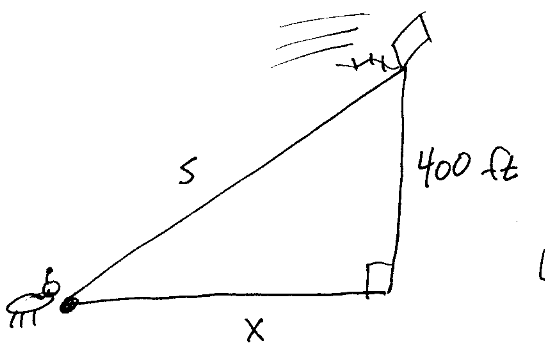
d) If $f'(x) = g'(x)$ for every x, then $f(x) = g(x)$ for every x.

False. Take $f(x) = x$ and $g(x) = x+1$. Then $f'(x) = g'(x) = 1$,
but clearly $f(x) \neq g(x)$.

e) If $f(x) = g(x)$ for every x, then $f'(x) = g'(x)$ for every x.

True.

8. (12 points) Uncle Ant is flying a kite. The kite is at a constant altitude of 400 feet, being blown away from Uncle Ant horizontally at a constant speed. If it's directly over his head at 9:00 am, and it's moved 800 feet horizontally by 1:00 pm, how fast is the string being payed out at the time when 500 feet of string is already out? (Assume that the string is perfectly taut, so that it forms a straight line, and that Uncle Ant's height is 0.)



s = length of string
x = horizontal distance

We're interested in $s'(t)$.

We have $x^2 + (400)^2 = s^2$, so $2x\,x' = 2s\,s'$.

At the time we're interested in, $s = 500$, so we have

$x^2 + 160{,}000 = 250{,}000$, or $x = 300$.

What's x'? Well, it's a constant. $x = 0$ at 9:00 am, and 4 hrs later $x = 800$, so $x' = \frac{800 - 0}{4} = 200$.

So: $2x\,x' = 2s\,s'$

$2 \cdot 300 \cdot 200 = 2 \cdot 500 \cdot s'$

$120 = s'$

ft/hr

9. (10 points) DO ONLY ONE of the following problems. I will grade only one of them, so if you try both, be sure to tell me which one you want graded.

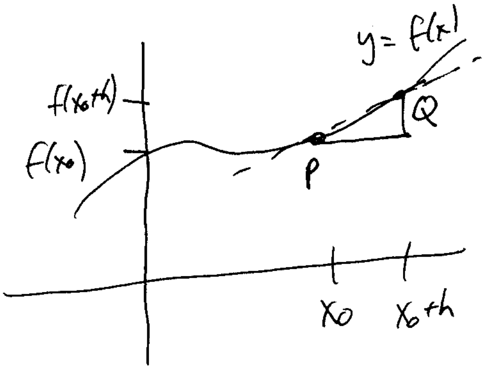
a) Prove the quotient rule. (You may use any other rules that we've learned.)

b) State the definition of the derivative of the function *f(x)* at the point x_0. Using a picture, explain (in words) why the slope of the tangent line to the graph *y=f(x)* at the point x_0 is given by that definition. Be sure to label your picture well.

a) $$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'$$

$$= \frac{f'}{g} + f \cdot \frac{-1}{g^2} \cdot g' = \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}$$

b) $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



The derivative at x_0 is equal to the slope of the tangent line to the graph $y = f(x)$ at the point P. The tangent line is the limit of the secant lines joining P & Q as Q gets closer & closer to P, i.e., as h goes to 0. Thus the slope of the tangent line is equal to the limit of the slopes of the secant lines as $h \to 0$, that is,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$