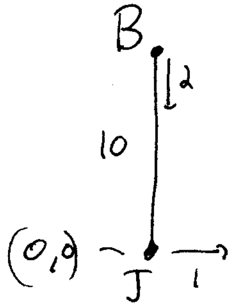
1. (20 points) The Batmobile is 10 miles due north of the Joker's Jetcycle. If Batman goes south at a constant speed of 2 miles/minute, and the Joker heads east at 1 mile/minute, what is the closest that Batman comes to the Joker? (Don't worry if this minimum distance is greater than zero – if Batman doesn't catch him, Robin the Boy Wonder will.)



Let's say that the Joker starts at the point $(0,0)$.
Then at time t, the Joker is at the point $(t,0)$.
At time t, Batman is at the point ~~(10-2t)~~ $(0, 10-2t)$.

So the distance between the two at time t is given by $d(t) = \sqrt{(t-0)^2 + (0-(10-2t))^2} = \sqrt{t^2 + (10-2t)^2}$

It'll be easier to find the critical pts. ~~minimum~~ of the distance squared,

$$d^2(t) = t^2 + (10-2t)^2$$

Find the critical pts: $\frac{d}{dt}\left(d^2(t)\right) = 2t + 2(10-2t)(-2)$

$$= 2t - 40 + 8t$$

$$= 10t - 40$$

This exists for all t, so the only critical pts. are where it's equal to 0

$$10t - 40 = 0$$
$$10t = 40$$
$$t = 4$$

Our endpoints are 0 & ∞.

At $t=0$, the distance is 10.

At $t=4$, the distance is $\sqrt{4^2 + (10-8)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ (b/w 4 & 5)

As $t \to \infty$, the distance also goes to ∞.

Thus the minimum distance is $2\sqrt{5}$.

2. (10 points) Uncle Ant is taking a walk. His position at time t is given by the equation $r(t) = (2\cos t, 3\sin t)$. Find an equation for the tangent line to Uncle Ant's path at the point corresponding to time $t = \pi/4$.

$r(t) = (x(t), y(t)) = (2\cos t, 3\sin t)$

The slope of the tangent line at $r(t)$ is $\dfrac{y'(t)}{x'(t)} = \dfrac{3\cos t}{-2\sin t}$.

So at $r(\frac{\pi}{4})$, the slope is $\dfrac{3\cos\frac{\pi}{4}}{-2\sin\frac{\pi}{4}} = \dfrac{3\cdot\frac{1}{\sqrt{2}}}{-2\cdot\frac{1}{\sqrt{2}}} = \dfrac{-3}{2}$

The point $r(\frac{\pi}{4})$ is $(2\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}) = (2\cdot\frac{1}{\sqrt{2}}, 3\cdot\frac{1}{\sqrt{2}}) = (\frac{2}{\sqrt{2}}, \frac{3}{\sqrt{2}})$

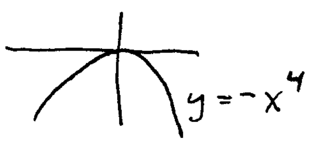
So the tangent line to the curve at the point $r(\frac{\pi}{4}) = (\frac{2}{\sqrt{2}}, \frac{3}{\sqrt{2}})$

is $y - \frac{3}{\sqrt{2}} = \frac{-3}{2}(x - \frac{2}{\sqrt{2}})$

3. (12 points) State whether each of the following is true or false. If it's false, give an example showing that it's false.

a) If x_0 is a critical point of f but $f''(x_0) = 0$, then x_0 is not a local maximum of f.

False: If $f(x) = -x^4$, then $f'(0) = f''(0) = 0$, but 0 is a local max.



b) If x_0 is not a local minimum of f, then x_0 is not a critical point of f.

False: Same example as part a). 0 is not a local min, but it is a critical pt.

c) If $f''(x_0) = 0$, then the graph of f has an inflection point at x_0.

False: Same example as part a). $f''(0) = 0$, but the second derivative is positive on both sides of 0. (An inflection pt. is where the second derivative changes sign.)

d) Economics is good.

True.

4. (10 points) For each of the following, compute the indicated limit, or state that it does not exist.

a) $\lim_{x\to\infty} \frac{e^{x^2}}{x}$

b) $\lim_{x\to 1} \frac{x^2}{x^2-2x+1}$

a) "plug in" ∞, get $\frac{\infty}{\infty}$.

Use L'Hopital's rule:

$\lim_{x\to\infty} \frac{e^{x^2}}{x} = \lim_{x\to\infty} \frac{2xe^{x^2}}{1}$ "=" ∞

The limit doesn't exist - the function just gets bigger and bigger.

b) plug in 1, get $\frac{1}{1-2+1} = \frac{1}{0}$

The limit doesn't exist.

5. (16 points) Use linear approximation to estimate $\sqrt[3]{28}$.

$f(x) \approx f(a) + f'(a)(x-a)$ Use $f(x) = x^{\frac{1}{3}}$ and $a = 27$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$f'(a) = \frac{1}{3}(27)^{-\frac{2}{3}} = \frac{1}{3}3^{-2} = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}$

Then $f(28) \approx f(27) + f'(27)(28-27)$

$\sqrt[3]{28} \approx 3 + \frac{1}{27}(1)$

$= 3 + \frac{1}{27}$

6. (20 points) Define the function f by $f(x) = x^{\frac{2}{3}} + \frac{2}{3}x$.

a) Find the critical points of f, or state that there are none.

b) Find the maximum and minimum values of f on the interval $-1 \le x \le 1$.

a) Critical pts. are where f' is zero or doesn't exist.

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3} = \frac{2}{3} \cdot \frac{1}{x^{\frac{1}{3}}} + \frac{2}{3}$$

This doesn't exist when $x=0$, since $\frac{1}{0^{\frac{1}{3}}} = \frac{1}{0}$ doesn't exist.

So 0 is a critical pt.

Now set $f'(x)=0$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3} = 0$$

$$x^{-\frac{1}{3}} = -1 \quad \text{cube each side}$$

$$x^{-1} = -1$$

$$\frac{1}{x} = -1$$

$$x = -1$$

So the crit. pts. are 0 & -1

b) Check the values of f on the critical pts. in the interval (namely 0 & -1) and the endpts, -1 & 1.

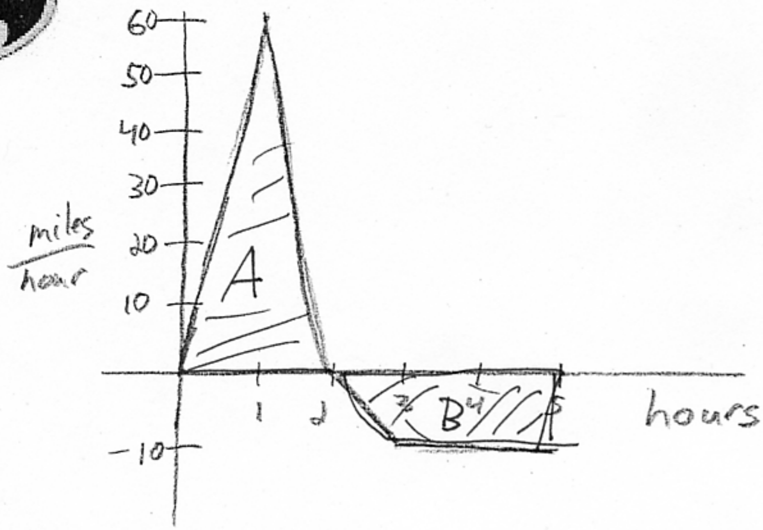
0: $f(0) = 0$

-1: $f(-1) = (-1)^{\frac{2}{3}} + \frac{2}{3}(-1) = (-1)^2 - \frac{2}{3} = 1 - \frac{2}{3} = \frac{1}{3}$

1: $f(1) = 1^{\frac{2}{3}} + \frac{2}{3} = 1 + \frac{2}{3} = \frac{5}{3}$

So the max. value is $\frac{5}{3}$ & the min value is 0.

7. (12 points) The figure below gives Batman's velocity during a trip starting from the Batcave. Positive velocities take him away from the Batcave and negative velocities take him toward it. Where is he at the end of five hours? When is he farthest from the Batcave? How far away is he at that time?



Distance is the integral of velocity, so after 5 hrs. he's $\int_0^5 v(x)\,dx$ miles from the Batcave (where v is his velocity).

$$\int_0^5 v(x)\,dx = A - B$$
$$= \tfrac{1}{2}\cdot 2\cdot 60 - \left(\tfrac{1}{2}\cdot 10\cdot 1 + 2\cdot 10\right)$$
$$= 60 - (5 + 20)$$
$$= 35 \text{ miles from the Batcave.}$$

He is farthest from the Batcave at 2 hours, when his distance is $\int_0^2 v(x)\,dx = A = 60$ miles.