Math 30 – Solns. to Practice 2nd Midterm Spring 2002

① $y(x) = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

Plug in: $y'' + kxy = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + kx \sum_{n=0}^{\infty} a_n x^n = 0$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + k \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$2 \cdot 1 a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} k a_{n-1} x^n = 0$$

$$2 a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + k a_{n-1} \right] x^n = 0$$

So $2a_2 = 0$, so $a_2 = 0$. Also, for $n \geq 1$, $(n+2)(n+1) a_{n+2} + k a_{n-1} = 0$

$$a_{n+2} = \frac{-k a_{n-1}}{(n+2)(n+1)}$$

Since $y(0) = 0$, $a_0 = 0$. Since $y'(0) = 1$, $a_1 = 1$.

So: $a_3 = \frac{-k a_0}{3 \cdot 2} = 0 \qquad a_6 = \frac{-k a_3}{6 \cdot 5} = 0$

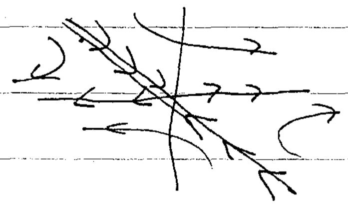
$a_4 = \frac{-k a_1}{4 \cdot 3} = \frac{-k}{4 \cdot 3} \qquad a_7 = \frac{-k a_4}{7 \cdot 6} = \frac{k^2}{7 \cdot 6 \cdot 4 \cdot 3}$

$a_5 = \frac{-k a_2}{5 \cdot 4} = 0 \qquad a_8 = \frac{-k a_5}{8 \cdot 7} = 0$

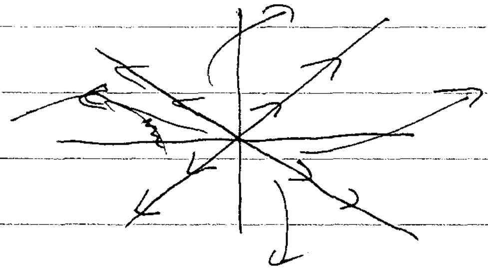
So $a_n = 0$ unless $n = 3m+1$, and

$$y(x) = 1 + \sum_{m=1}^{\infty} \frac{(-k)^m x^{3m+1}}{(3m+1)(3m) \cdot (3m-2)(3m-3) \cdots (7)(6)(4)(3)}$$

(2) a) $\vec{x}' = \begin{pmatrix}1 & 5\\ 0 & -3\end{pmatrix}\vec{x}$. Eigenvectors are $\begin{pmatrix}1\\0\end{pmatrix}$, $\lambda_1 = 1$, & $\begin{pmatrix}-5\\4\end{pmatrix}$, $\lambda_2 = -3$.
So $(0,0)$ is a saddle:



b) $\vec{x}' = \begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix}\vec{x}$. Eigenvectors are $\begin{pmatrix}-1\\1\end{pmatrix}$, $\lambda_1 = 3$, & $\begin{pmatrix}1\\1\end{pmatrix}$, $\lambda_2 = 5$.
So $(0,0)$ is a source:



c) $\vec{x}' = \begin{pmatrix}2 & -3\\ 5 & -4\end{pmatrix}$. Eigenvalues are $-1 \pm i\sqrt{6}$, so $(0,0)$ is a spiral sink.

(3) a) Char. poly. is $r^4 - 4r^3 + 4r^2 = r^2(r^2-4r+4) = r^2(r-2)^2$. The roots are 0 (twice) and 2 (twice), so the soln. is $c_1 + c_2 t + c_3 e^{2t} + c_4 t e^{2t}$

b) First, solve homogeneous: Char. poly. is $r^3 + r^2 + r + 1 = (r^2+1)(r+1)$. The roots are $-1, i, -i$, so the gen. soln. is $c_1 e^{-t} + c_2 \cos t + c_3 \sin t$.
To find a particular soln, guess $v(t) = b$ — get $b = 7$. So the gen. soln. is $c_1 e^{-t} + c_2 \cos t + c_3 \sin t + 7$.

(4) $(s^2 - 2s + 2)\,Y(s) = \frac{s}{s^2+1}$, $Y(s) = \frac{s}{(s^2+1)(s^2-2s+2)} = \frac{\frac{1}{5}s - \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s + \frac{4}{5}}{(s-1)^2+1}$

$$Y(s) = \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1} - \frac{1}{5}\frac{s}{(s-1)^2+1} + \frac{4}{5}\frac{1}{(s-1)^2+1}$$

$$y(t) = \frac{1}{5}\cos t - \frac{2}{5}\sin t - \frac{1}{5}e^{t}\cos t + \frac{4}{5}e^{t}\sin t$$

5) a) i) $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ ii) $\det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix} = 0$, so no inverse

b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 6 & 0 & 0 \\ 1 & 1 & -1 & 9 \\ 3 & 3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix} = 10\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 1.

6) The soln. is $e^{Pt}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{t} + \frac{1}{2}e^{3t} - \frac{1}{2}e^{5t} \\ e^{3t} - e^{5t} \\ -e^{5t} \end{pmatrix}$

7) Use Theorem 5.3.1 on p. 250. Since $\sin x$ and $(4x^2 - x)$ are analytic everywhere (ie, radius of conv = "∞"), the series soln. also converges everywhere.

8) $y' = v$, $y'' = v' = w$, $y''' = w' = u$, and $y^{(4)} = u' = \frac{1}{3}(5y'' - 9y + 6e^{t} - 2t)$
$= \frac{1}{3}(5w - 9y + 6e^{t} - 2t)$,

or $y' = v$
$v' = w$
$w' = u$
$u' = \frac{1}{3}(5w - 9y + 6e^{t} - 2t)$, or $\begin{pmatrix} y \\ v \\ w \\ u \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & \frac{5}{3} & 0 \end{pmatrix}\begin{pmatrix} y \\ v \\ w \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6e^{t} - 2t \end{pmatrix}$

9) a) $c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}e^{-4t} + \begin{pmatrix} -\frac{3}{7}e^{3t} - \frac{1}{2}t \\ -\frac{2}{7}e^{3t} - \frac{1}{16} \end{pmatrix}$

(use undet. coeffs.)

b) $c_1\begin{pmatrix} \cos 2t \\ \cos 2t + 2\sin 2t \end{pmatrix}e^{5t} + c_2\begin{pmatrix} \sin 2t \\ -2\cos 2t + \sin 2t \end{pmatrix}e^{5t}$

10) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. ~~Soln is 4000 5000~~

Soln is $5000e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 3000e^{5t}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

In the long run, the second term dominates, and the ratio is 3 : 4.