(1) (10 points) (*This question is for section 1 (without linear algebra) only. If you're in section 2, you've got the wrong version of the exam.*) The vector $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ is an eigenvector for the matrix $\mathbf{P}$ with eigenvalue 3. What is $e^{\mathbf{P}}\mathbf{x}$?

Since $P\vec{x} = 3\vec{x}$, $e^{P}\vec{x} = e^{3}\vec{x} = \begin{pmatrix} e^3 \\ 0 \\ 2e^3 \end{pmatrix}$

(2) (10 points) Find the general solution of the ODE $y'' - 3y' + 2y = 3e^t$.

1st, solve the homogeneous version: $y'' - 3y' + 2y = 0$. The char. poly. is $r^2 - 3r + 2 = (r-2)(r-1)$. So the soln is $C_1 e^{2t} + C_2 e^{t}$.

Now, guess a particular soln. We would guess ae^t, but that's a soln. to the homog. So we guess ate^t:

$$(ate^t)' = ae^t + ate^t, \quad (ate^t)'' = 2ae^t + ate^t$$

So $y'' - 3y' + 2y = 3e^t$ becomes

$$2ae^t + ate^t - 3(ae^t + ate^t) + 2ate^t = 3e^t$$

$$-ae^t = 3e^t$$

$$a = -3$$

So $-3te^t$ is a particular soln., and the general soln. is $C_1 e^{2t} + C_2 e^t - 3te^t$

(10 points) (*This question is for section 2 (with linear algebra) only. If you're in section 1, you've got the wrong version of the exam.*) True or false: When numerically integrating a vector field using Runge-Kutta, we can always get better accuracy by decreasing the step size (i.e., by making Δt smaller). Explain.

False: If our step size gets too small, we start to have problems with roundoff error, so that a smaller step size can actually decrease the accuracy.

(10 points) Find the general solution of the ODE $y'' - 3y' + 2y = 3e^t$.

(3) (15 points) Solve the ODE $\frac{dy}{dt} = ry - ky^2$, $r > 0$ and $k > 0$. (HINT: Make the substitution $v = 1/y$.)

$v = \frac{1}{y}$, or $y = \frac{1}{v}$ ($y \equiv 0$ is an equilibrium soln.)

$y' = \frac{-v'}{v^2}$

So the ODE becomes $\frac{-v'}{v^2} = \frac{r}{v} - \frac{k}{v^2}$ - multiply through by v^2

$-v' = rv - k$, or $v' + rv = k$.

This is linear - the integrating factor is $e^{\int r\,dt} = e^{rt}$. We set

$(ve^{rt})' = ke^{rt}$, or $ve^{rt} = \frac{k}{r}e^{rt} + C$, or

$v = \frac{k}{r} + Ce^{-rt}$. Then $y = \frac{1}{v} = \frac{1}{\frac{k}{r} + Ce^{-rt}} = \frac{r}{k + Cre^{-rt}}$

OR $y \equiv 0$.

(4) (15 points) Find the general solution of the system $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{x}$.

Eigenvalue is -2 only. Find eigenvectors: solve $\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$,

$\begin{pmatrix} -2x + y \\ -2y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix} \Rightarrow y = 0$. Simplest: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. So one soln. is $e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Another soln. is $e^{-2t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}\right)$, where $\begin{pmatrix} x \\ y \end{pmatrix}$ is a generalized eigenvector, ie, $\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -2x + y \\ -2y \end{pmatrix} = \begin{pmatrix} -2x + 1 \\ -2y \end{pmatrix} \Rightarrow y = 1$. Simplest: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So the general soln is $C_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-2t}\left(t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$$= \begin{pmatrix} (C_1 + C_2 t)e^{-2t} \\ C_2 e^{-2t} \end{pmatrix}$$

(5) (20 points) Suppose two similar countries Y and Z are engaged in an arms race. Let $y(t)$ and $z(t)$ denote the size of the stockpiles of arms of Y and Z, respectively. We model this situation with the system of differential equations

$$\begin{aligned} y' &= h(y,z) \\ z' &= k(y,z) \end{aligned}.$$

Suppose that all we know about the functions h and k are the two assumptions:

(i) If country Z's stockpile of arms is not changing, then any increase in the size of Y's stockpile of arms results in a decrease in the rate of arms building in country Y. Similarly, if country Y's stockpile of arms is not changing, then any increase in the size of Z's stockpile of arms results in a decrease in the rate of arms building in country Z.

(ii) If either country increases its stockpile, the other responds by increasing its rate of arms production.

(a) What do the assumptions imply about $\partial h/\partial y$ and $\partial k/\partial z$?

$y \nearrow \Rightarrow y' \searrow$, so $\frac{\partial h}{\partial y} < 0$.

Similarly, $z \nearrow \Rightarrow z' \searrow$, so $\frac{\partial k}{\partial z} < 0$

(b) What do the assumptions imply about $\partial h/\partial z$ and $\partial k/\partial y$?

$z \nearrow \Rightarrow y' \nearrow$, so $\frac{\partial h}{\partial z} > 0$

Similarly, $y \nearrow \Rightarrow z' \nearrow$, so $\frac{\partial k}{\partial y} > 0$

(c) What types of equilibrium points are possible for this system? Justify your answer.

To classify an eq. pt: linearize. The derivative matrix is

$$\begin{pmatrix} \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \\ \frac{\partial k}{\partial y} & \frac{\partial k}{\partial z} \end{pmatrix} = \begin{pmatrix} -A & B \\ C & -D \end{pmatrix}, \text{ where } A, B, C, D > 0.$$

What can the eigenvalues be? The char. poly. is $\det\begin{pmatrix} -A-\lambda & B \\ C & -D-\lambda \end{pmatrix}$

$= \lambda^2 + (A+D)\lambda + (AD-BC)$. The eigenvalues are the roots,

$-(A+D) \pm \sqrt{(A+D)^2 - 4(AD-BC)}$. Depending on what $A, B, C, \&\ D$ are,

we can get: 2 negative eigenvalues (eq. pt. is a <u>sink</u>), or
1 pos. & 1 neg. eigenvalue (eq. pt. is a <u>saddle</u>), or
2 complex eigenvalues with neg. real part (eq. pt. is a <u>spiral sink</u>)

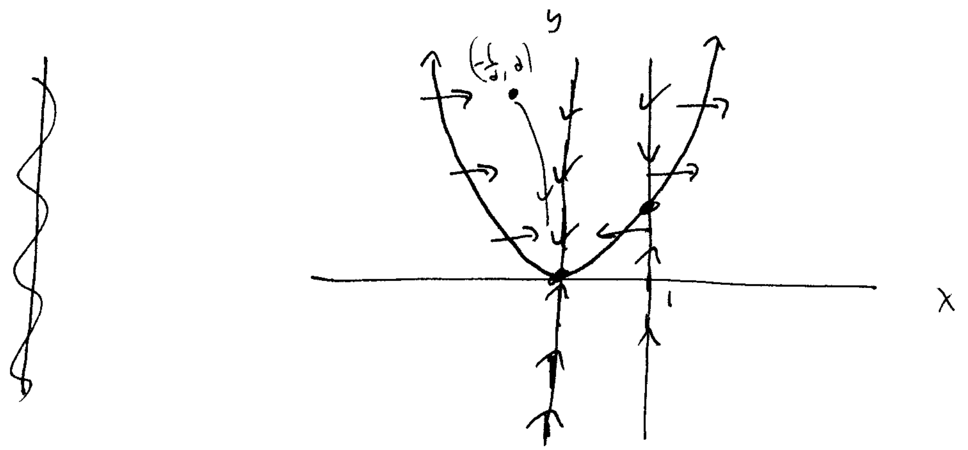
(6) (15 points) Consider the system

$$x' = x(x-1)$$
$$y' = x^2 - y$$

Sketch the x- and y-nullclines. Then find all equilibrium points. Using the direction of the vector field between the nullclines, describe the possible behavior of the solution corresponding to the initial condition $x(0) = -0.5$, $y(0) = 2$. (N.B.: Look at the whole plane, *not* just the first quadrant.)

x-nullcline: $x' = 0 = x(x-1)$ $\quad x=0$ & $x=1$

y-nullcline: $y' = 0 = x^2 - y$ $\quad y = x^2$



eq. pts.: $(0,0)$ & $(1,1)$

The soln. starting at $(-\frac{1}{2}, 2)$ heads down & right towards $(0,0)$.

(7) (10 points) Consider the initial value problem $y'' + y = 0$, $y(0) = y(a) = 0$. For what values of a (if any) will there be more than one solution to the IVP?

The general soln. is $c_1 \cos t + c_2 \sin t$.

$y(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

So $y = c_2 \sin t$. Plug in $y(a) = 0$: $c_2 \sin a = 0$, so either $c_2 = 0$ or $\sin a = 0$. Thus if $a = n\pi$ for some integer n, then $c_2 \sin t$ is a soln. for <u>any</u> constant c_2, so there are infinitely many solns. If $a \neq n\pi$, then the only soln. is $y \equiv 0$.

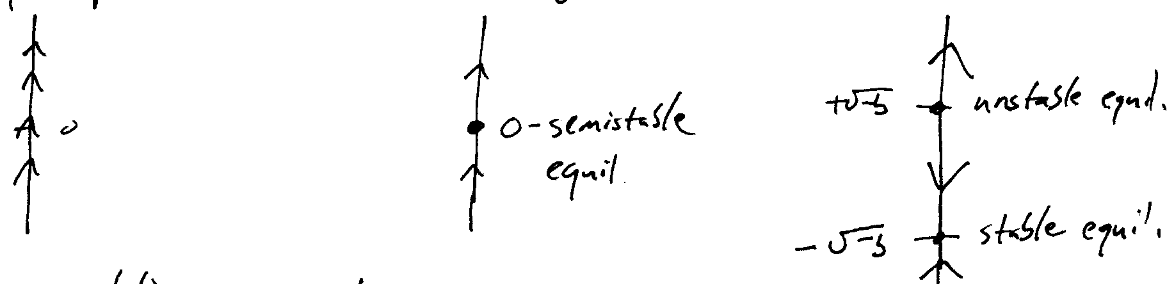
(8) (15 points) Let $y(t)$ be the population of fish in a certain lake at time t. Assume that the population is governed by the ODE $y' = y^2 + b$ (where b is a constant) and that initially there are 1000 fish in the lake. What's the long-term behavior of the fish population? (HINT: Your answer will be different for different values of b.)

Look at the phase line:

$b > 0$: $y^2 + b$ is never 0 no equil. pts.

$b = 0$: $y^2 + b = 0 \Rightarrow y = 0$

$b < 0$: $y^2 + b = 0 \Rightarrow y = \pm\sqrt{-b}$



If $b \geq 0$: population goes to ∞.

If $b < 0$ but $1000 > \sqrt{-b}$, ie, $b > -1{,}000{,}000$, population still goes to ∞.

If $b = -1{,}000{,}000$, then $y = 1000$ is an equil. soln. A slight increase in pop. sends the pop. to ∞; a slight decrease to 0.

If $b < -1{,}000{,}000$, then the pop. decreases to 0.

9) (20 points) Solve the following ODEs:

(a) $y' = (ty)^2$. (Find the general solution.)

$\frac{dy}{dt} = t^2 y^2$: separate variables. $y=0$ is an equil. soln. Otherwise,

$\frac{1}{y^2}\,dy = t^2\,dt$, $\frac{-1}{y} = \frac{1}{3}t^3 + C$

$$y = \frac{-1}{\frac{1}{3}t^3 + C} \quad \text{or } y = 0$$

(b) $x + yy' = 0$, $y(0) = -2$. For what x values does the solution exist?

This is exact: $\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(y)$.

$(x + yy') = \left(\frac{x^2+y^2}{2}\right)'$, so we have $\left(\frac{x^2+y^2}{2}\right)' = 0$,

$\frac{x^2+y^2}{2} = C$. Since $y(0) = -2$, $\frac{0+4}{2} = C$, $C = 2$.

Thus $\frac{x^2+y^2}{2} = 2$, or $x^2+y^2 = 4$. Since $y(0) = -2$, the soln

is $y = -\sqrt{4-x^2}$, which exists for $-2 \le x \le 2$.

(10) (15 points) Suppose that the solution $y(t)$ of the IVP $y'' + py' + qy = \delta(t)$, $y(0) = y'(0) = 0$ (where p and q are constants) has a Laplace transform $\mathfrak{L}\{y(t)\}$ whose value at $s = 0$ is $1/5$ and whose value at $s = 2$ is $1/17$.

(a) Find p and q.

$$\mathcal{L}(y''+py'+qy) = \mathcal{L}(\delta(t))$$

$$s^2 Y(s) + ps\,Y(s) + q\,Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+ps+q}$$

Plug in: $Y(0) = \frac{1}{5} = \frac{1}{0+0+q}$, so $q = 5$.

$$Y(2) = \frac{1}{4+2p+5} = \frac{1}{17}, \text{ so } p = 4$$

(b) Find $y(t)$. (If you couldn't solve part (a), use the values $p = 2$ and $q = 5$.)

$$Y(s) = \frac{1}{s^2+4s+5} \quad \text{- complete the square}$$

$$= \frac{1}{(s^2+4s+4)+1} = \frac{1}{(s+2)^2+1}$$

$$\text{So } y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2+1}\right) = e^{-2t}\sin t$$

(If you used 2 & 5: $Y(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s^2+2s+1)+4} = \frac{1}{(s+1)^2+4}$

$$\text{So } y(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\cdot\frac{2}{(s+1)^2+4}\right)$$

$$= \frac{1}{2}e^{-t}\sin 2t$$)

(11) (15 points) Find the first four nonzero terms of the series solution of Airy's equation, $y'' = xy$, with initial conditions $y(0) = 1$, $y'(0) = 0$.

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

$$y'' = xy$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) \cancel{} a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Since there is no constant term on the right, $2a_2 = 0$.

Furthermore, $y(0) = a_0$, so $a_0 = 1$, and $y'(0) = a_1$, so $a_1 = 0$.

We have, for $n \geq 1$, that $(n+2)(n+1) a_{n+2} = a_{n-1}$

or $\quad a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$

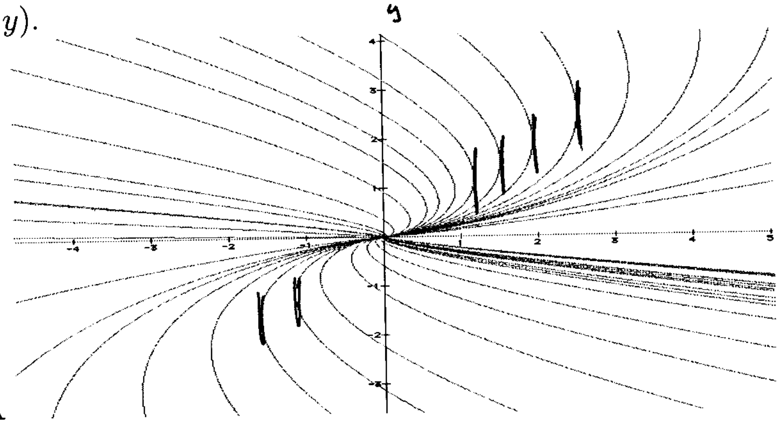
Thus $a_3 = \dfrac{a_0}{3 \cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{a_1}{4 \cdot 3} = 0$, $a_5 = \dfrac{a_2}{5 \cdot 4} = 0$,

$a_6 = \dfrac{a_3}{6 \cdot 5} = \dfrac{1}{180}$, $a_7 = \dfrac{a_4}{7 \cdot 6} = 0$, $a_8 = \dfrac{a_5}{8 \cdot 7} = 0$,

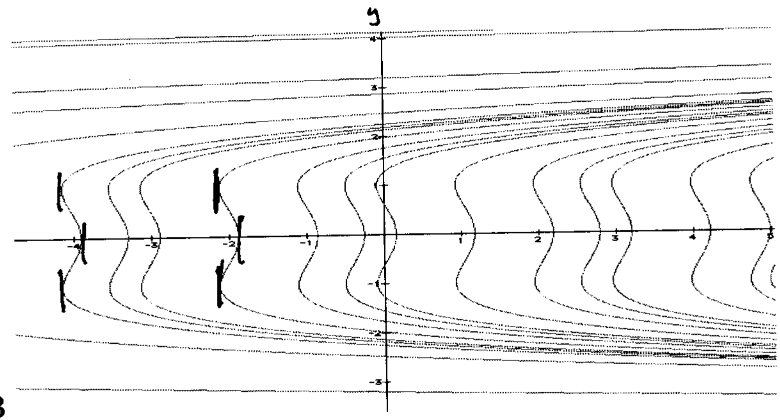
$a_9 = \dfrac{a_6}{9 \cdot 8} = \dfrac{1}{9 \cdot 8 \cdot 180}$

So our soln. is $\quad 1 + \frac{1}{6} x^3 + \frac{1}{180} x^6 + \dfrac{1}{9 \cdot 8 \cdot 180} x^9 + \dots$

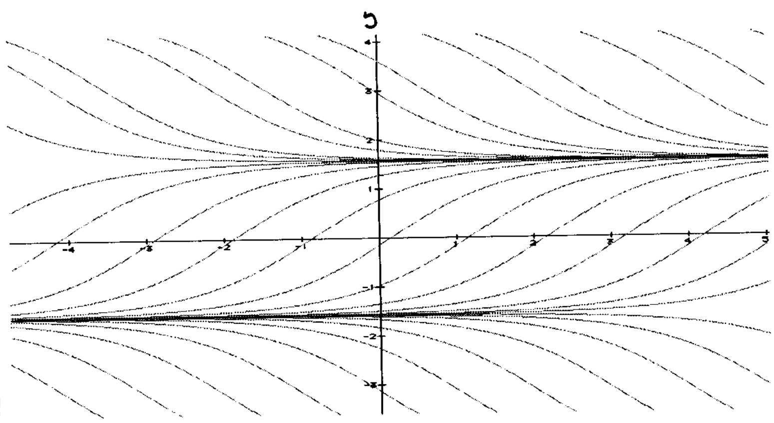
(12) (20 points) The figures below show solutions in the ty-plane of equations of the form $y' = F(t, y)$.



A



B



C

(a) Which systems, if any, have some solutions which are not unique?

A & B – for example, at places marked above

(b) Which systems, if any, are autonomous? i.e., y' depends only on y, meaning that the picture looks the same if you shift it left or right (the t direction): B & C

(13) (20 points) A 30-gallon tank initially contains 15 gallons of saltwater containing 6 pounds of salt. Suppose saltwater containing 1 pound of salt per gallon is pumped into the top of the tank at the rate of 2 gallons per minute, while a well-mixed solution leaves the bottom of the tank at the rate of 1 gallon per minute. How much salt is in the tank when the tank is full?

Let $y(t)$ = amt. of salt at time t. The amt. of salt in one gallon is $\frac{y(t)}{\#\text{ of gallons}}$. For $0 \le t \le 15$, the # of gallons is $15+t$, so there is $\frac{y(t)}{15+t}$ lbs. of salt in one gallon. Thus the differential eqn. is

$y'(t) = 2 - \frac{y(t)}{15+t}$. This is linear: $y' + \frac{y}{15+t} = 2$. The integrating factor is $e^{\int \frac{1}{15+t}} = e^{\ln|15+t|} = 15+t$ (since $t \ge 0$). The eqn. becomes

$(15+t)y' + y = 30+2t$, or $\big((15+t)y\big)' = 30+2t$, or

$(15+t)y = 30t + t^2 + C$. Plug in $y(0) = 6$: $15 \cdot 6 = C$, or $C = 90$.

The tank fills up when $t = 15$, so

$$30y(15) = 30 \cdot 15 + 15^2 + 90$$
$$30y(15) = 765$$
$$y(15) = 765/30$$

EXTRA CREDIT (5 points) True or false: Classification of mathematical problems as linear and nonlinear is like classification of the universe as bananas and non-bananas.