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(1) Solve the following differential equations. (Give the general solution if no initial condition is specified.)

8 (a) $t^5y' + y^5 = 0$ Separate variables:

$$t^5 \frac{dy}{dt} = -y^5$$

$$\frac{1}{y^5}\,dy = \frac{-1}{t^5}\,dt$$

$$-\frac{1}{4}\cdot\frac{1}{y^4} = \frac{1}{4}\frac{1}{t^4} + C$$

or $\boxed{\frac{1}{t^4} + \frac{1}{y^4} = D}$

(b) $e^{2y} - y\cos(ty) + (2te^{2y} - t\cos(ty) + 2y)y' = 0$

This is exact. It's the same as

$$\left(te^{2y} - \sin(ty) + y^2\right)' = 0,$$

so $\boxed{te^{2y} - \sin(ty) + y^2 = C}$

(c) $\dfrac{dy}{dx} + 2xy = x$, $y(0) = -3$

This is linear. The integrating factor $\mu(x)$ is e^{x^2}

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}$$

$$(e^{x^2}y)' = xe^{x^2}$$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C$$

$$y = \frac{1}{2} + Ce^{-x^2}$$

$$y(0) = -3 = \frac{1}{2} + C$$

$$-\frac{7}{2} = C$$

so $\boxed{y = \frac{1}{2} - \frac{7}{2}e^{-x^2}}$

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(2) Find the general solution to the ODE $y'' - 2y' + y = e^{-t}$.

1st, solve the assoc. homogeneous ODE

$$y'' - 2y' + y = 0$$

The char. poly. is $r^2 - 2r + 1$, which has the single root $r = 1$.

Thus the solns. are e^t & te^t.

To find a particular soln, guess $y = ae^{-t}$

and solve $y'' - 2y' + y = e^{-t}$

$$ae^{-t} + 2ae^{-t} + ae^{-t} = e^{-t}$$

$$4ae^{-t} = e^{-t}$$

$$a = \frac{1}{4}$$

So a particular soln. is $y = \frac{1}{4}e^{-t}$,

and the general soln. is

$$\boxed{c_1 e^t + c_2 te^t + \frac{1}{4}e^{-t}}$$

) Consider the initial value problem $y'' - 3y' + 2y = 0$, $y(0) = y_0$, $y'(0) = y_0'$. For what initial conditions (i.e., what values of y_0 and y_0') will the solution tend to 0 as $t \to \infty$?

The char. poly. is $r^2 - 3r + 2$, which has roots $r=1$ & $r=2$. So the general soln. is $c_1 e^t + c_2 e^{2t}$, which will tend to 0 only if $c_1 = c_2 = 0$, ie, only if $y_0 = y_0' = 0$.

(4) Newton's law of cooling states that the rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature in the body and the constant temperature T_m of the surrounding medium. That is, $\frac{dT}{dt} = k(T - T_m)$. Yesterday, when I took my delicious cake out of the oven, I measured its temperature to be 300°F. Three minutes later its temperature was 200°F. How long will it take to cool off to 80°F if the room temperature is a balmy 70°F?

$T' = k(T - T_m) = k(T - 70)$, $T(0) = 300$, $T(3) = 200$

Separate variables: $\frac{dT}{T-70} = k\,dt$

$\ln|T - 70| = kt + C$

or $T = 70 + De^{kt}$

Since $T(0) = 300$, $D = 230$.

Also, $T(3) = 200 = 70 + 230e^{3k}$

$130 = 230e^{3k}$

$\frac{1}{3}\ln\frac{13}{23} = k$

So $T(t) = 70 + 230e^{\left(\frac{1}{3}\ln\frac{13}{23}\right)t}$.

Solve $T(t) = 80$ for t:

$70 + 230e^{\frac{1}{3}\ln\frac{13}{23}t} = 80$

$230e^{\frac{1}{3}\ln\frac{13}{23}t} = 10$

$e^{\frac{1}{3}\ln\frac{13}{23}t} = \frac{1}{23}$

$\frac{1}{3}\ln\frac{13}{23}\,t = \ln\frac{1}{23}$

$$\boxed{t = \frac{3\ln\frac{1}{23}}{\ln\frac{13}{23}}}$$

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(5) Discuss the differences between linear homogeneous and linear inhomogeneous ODEs.

Linear homogeneous: - linear comb. of solns. is a solns.

- can find a fund. set of solns: $c_1 y_1 + c_2 y_2$
- 0 is always a soln

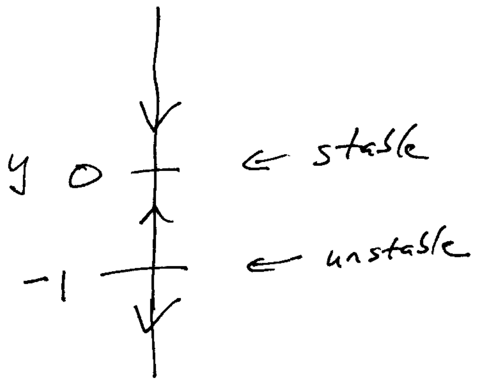
inhomogeneous - linear comb. of solns. is not a soln

- general soln. consists of soln. to homo. + particular soln
- 0 is not a soln.

15 +0

(6) Find the equilibria for the ODE $y' = -y^2 - y$. Are they stable, unstable, or semistable? What is the long-term behavior of the solutions?

$y' = -y(y+1)$ Equilibria are $y = 0, -1$



solns ~~-~~ increase to 0 if they begin between ~~...~~ -1 & 0

- decrease to 0 " " " above 0
- decrease to $-\infty$ " " " below -1.

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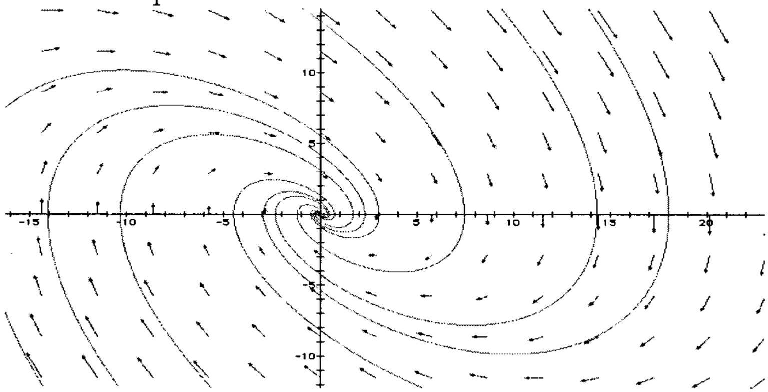
(7) The figure below shows the vector field $\frac{d}{dt}\left(\begin{bmatrix} y \\ y' \end{bmatrix}\right)$ associated to a differential equation $y'' = F(y, y')$, along with some solution curves. Discuss the long-term behavior of solutions to the differential equation.



All solns. go to 0.

EXTRA CREDIT (2 points) Would you vote for Senator Haddock? Why or why not?