(1) Find the general solution to the differential equation

$$y'' - y' - 6y = 12e^t + 36t.$$

1^{st}, solve the associated homogeneous: $y'' - y' - 6 = 0$. (✱)

Guess $e^{st} = y$ & get $s^2 e^{st} - se^{st} - 6e^{st} = 0$

$$s^2 - s - 6 = 0$$

$$(s-3)(s+2) = 0 \qquad s = 3, -2$$

So solns. to (✱) are e^{3t} & e^{-2t}.

Next, find a particular soln. Guess $Ke^t + At + B = y$, & get

$$Ke^t - (Ke^t + A) - 6(Ke^t + At + B) = 12e^t + 36t$$

$$-6Ke^t - 6At - A - 6B = 12e^t + 36t$$

So $-6K = 12$, $\quad -6A = 36$, $\quad$ & $-A - 6B = 0$

$K = -2 \qquad A = -6 \qquad -6B = -6$

$B = 1$

So a particular soln is $-2e^t - 6t + 1$,

and the general soln is

$$\boxed{c_1 e^{3t} + c_2 e^{-2t} - 2e^t - 6t + 1}$$

(2) Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2y - 2te^{t^2} \\ (1+t^2)x + ty + e^{t^2}(1+t^2) \end{pmatrix}. \qquad (*)$$

Both $\mathbf{Y}_1(t) = \begin{pmatrix} 0 \\ te^{t^2} \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} e^{t^2} \\ 2te^{t^2} \end{pmatrix}$ are solutions of $(*)$. Find the solution $\mathbf{Y}_0(t)$ of $(*)$ satisfying the initial condition $\mathbf{Y}_0(1) = \begin{pmatrix} -2e \\ -e \end{pmatrix}$.

This is a linear nonhomogeneous eqn: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2y \\ (1+t^2)x + ty \end{pmatrix} + \begin{pmatrix} -2te^{t^2} \\ e^{t^2}(1+t^2) \end{pmatrix}$.

So every soln. has the form $\bar{Y}(t) + Y_1(t)$, where $\bar{Y}(t)$ is ~~a soln~~ some soln. of the assoc. homogeneous.

We know that $Y_2 - Y_1 = \begin{pmatrix} e^{t^2} \\ te^{t^2} \end{pmatrix}$ is a soln. of the homogeneous,

so we try to find $Y_0 = c\begin{pmatrix} e^{t^2} \\ te^{t^2} \end{pmatrix} + Y_1$ satisfying $Y_0(1) = \begin{pmatrix} -2e \\ -e \end{pmatrix}$.

$$\begin{pmatrix} -2e \\ -e \end{pmatrix} = Y_0(1) = c\begin{pmatrix} e \\ e \end{pmatrix} + Y_1(1) = c\begin{pmatrix} e \\ e \end{pmatrix} + \begin{pmatrix} 0 \\ e \end{pmatrix}$$

$$\begin{pmatrix} -2e \\ -e \end{pmatrix} = \begin{pmatrix} c\cdot e \\ c\cdot e + e \end{pmatrix},$$ so $c = -2$ works.

So $Y_0(t) = -2\begin{pmatrix} e^{t^2} \\ te^{t^2} \end{pmatrix} + \begin{pmatrix} 0 \\ te^{t^2} \end{pmatrix} = \begin{pmatrix} -2e^{t^2} \\ -te^{t^2} \end{pmatrix}$.

(3) Let $\overline{y}(t)$ be the solution of the IVP

$$y'' + py' + qy = \delta_0(t); \qquad y(0) = 0,\ y'(0) = 0.$$

Find the solution to the IVP

$$y'' + py' + qy = f(t); \qquad y(0) = 0,\ y'(0) = a,$$

in terms of a, $f(t)$, and $\overline{y}(t)$.

We know that $\mathcal{L}[y''+py'+qy] = s^2 Y(s) - sy(0) - y'(0) + ps\,Y(s) - py(0) + qY(s)$

$$= (s^2+ps+q)Y(s) - sy(0) - y'(0) - py(0).$$

Plugging into the 1st IVP, we get

$$(s^2+ps+q)Y(s) = 1$$

$$Y(s) = \frac{1}{s^2+ps+q}$$

$$\overline{y}(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2+ps+q}\right]$$

Plugging into the 2nd IVP, we get

$$(s^2+ps+q)Y(s) - a = F(s)$$

$$Y(s) = \frac{a}{s^2+ps+q} + F(s)\cdot\frac{1}{s^2+ps+q}$$

$$\boxed{\text{so } y(t) = a\overline{y} + (f*\overline{y})(t)}$$

(4) Find the general solution of the system

$$\mathbf{Y}' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \mathbf{Y}.$$

Find the eigenvalues: $\det\begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = \lambda^2 - 8\lambda + 7 + 9 = 0$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

So $\lambda = 4$ is an eigenvalue of multiplicity 4.

Find the (single) assoc. eigenvector: $\begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$

$$x - 3y = 4x \Rightarrow y = -x$$

$$3x + 7y = 4y \Rightarrow y = -x$$

Simplest eigenvector: $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Next, find a generalized eigenvector: $\begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$x - 3y = 4x + 1 \Rightarrow -3y = 3x + 1$$

$$3x + 7y = 4y - 1 \Rightarrow 3x + 1 = -3y$$

simplest: $\vec{v}_0 = \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}$

So the general solution is $\boxed{c_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{4t}\left(\begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)}$

(5) (a) If $y(t) = \sum_{n=0}^{\infty} a_n t^n$ is the solution to the IVP

$$y'' + ty' + y = 0; \qquad y(0) = 1,\ y'(0) = 0,$$

find a_n for $n = 0, \ldots, 4$.

$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$. Since $y(0)=1$, $a_0 = 1$. Since $y'(0)=0$, $a_1 = 0$.

So $y(t) = 1 + a_2 t^2 + a_3 t^3 + \cdots$, $y'(t) = 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots$

$ty'(t) = 2a_2 t^2 + 3a_3 t^3 + 4a_4 t^4 + \cdots$

$y''(t) = \cancel{2a_2 + 9a_3 t^2 + 16 a_4 t^4}$

$2a_2 + 6a_3 t + 12 a_4 t^2$

So, $0 = y'' + ty' + y = (2a_2 + 1) + (6a_3)t + (12a_4 + 2a_2 + a_2)t^2 + \cdots$

So $0 = 2a_2 + 1$, $0 = 6a_3$, $0 = 12a_4 + 3a_2$

$a_2 = -\frac{1}{2}$ $\quad a_3 = 0$ $\quad -3a_2 = 12a_4$

$\frac{3}{2} = 12 a_4$

$\frac{1}{8} = a_4$

$\boxed{\text{So } a_0 = 1,\ a_1 = 0,\ a_2 = -\tfrac{1}{2},\ a_3 = 0,\ a_4 = \tfrac{1}{8}}$

(b) Determine the singular points of the differential equation

$$2t(t-2)^2 y'' + 3ty' + (t-2)y = 0$$

and classify them as regular or irregular.

Rewrite as $y'' + \frac{3y'}{2(t-2)^2} + \frac{1}{2t(t-2)}y = 0$

Singular pts. are $t=2$ & $t=0$. 0 is a regular sing. pt., since it is only a single root of the denominator of y's coefficient.

2 is irregular, since it's a double root of the denominator of (y')'s coefficient.

(6) What is the long-term behavior of solutions of the system

$$\mathbf{Y}' = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{Y}?$$

(NOTE: I didn't ask you to find the solutions themselves.)

Depends on the eigenvalues. $\det \begin{vmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2+2\lambda+1+2 = 0$

$\lambda^2+2\lambda+3=0$

$$\lambda = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}\, i}{2}$$

$$= -1 \pm \sqrt{2}\, i$$

Complex eigenvalues with negative real part $\Longrightarrow$ spiral sink.

So all solns. spiral in to the origin.

EXTRA CREDIT (5 pts) Explain why the solutions of a linear homogeneous system of ordin differential equations form a vector space.

$\vec{y}\,' = A(t)\vec{y}$, or $\vec{y}\,' - A(t)\vec{y} = 0$, or $\underbrace{\left[\frac{d}{dt} - A(t)\right]}_{L} \vec{y} = 0$

L is a linear operator, so its ~~soln~~ kernel (null space) is a vector space