Exam I
Math 18-01

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Write solutions on a separate sheet of paper. Explain all work. No calculators are allowed, and no numerical answer must be simplified.

1) The path $\mathbf{c}:\mathbb{R}\to\mathbb{R}^3$ is defined $\mathbf{c}(t)=t\mathbf{i}+e^t\mathbf{j}+\log t\mathbf{k}$, and the function $\mathbf{f}:\mathbb{R}^3\to\mathbb{R}^3$ is defined $\mathbf{f}(x,y,z)=\left(xy,x+y,z^2\right)$.
   a) Find an equation for the tangent line to the curve of the path $\mathbf{c}$ at $\mathbf{c}(1)$.
      $\ell(t)=\mathbf{i}+e\mathbf{j}+t\left(\mathbf{i}+e\mathbf{j}+\mathbf{k}\right)$
   b) Find $\mathbf{Df}(1,e,0)$.
      $$\left.\begin{bmatrix} y & x & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2z \end{bmatrix}\right|_{(1,e,0)} = \begin{bmatrix} e & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
   c) Find a tangent vector to the curve of the path $(\mathbf{f}\circ\mathbf{c})$ at $(\mathbf{f}\circ\mathbf{c})(1)$.
      $(\mathbf{f}\circ\mathbf{c})'(1)=\mathbf{Df}(1,e,0)\mathbf{Dc}=2e\mathbf{i}+(1+e)\mathbf{j}$
2) Find equations for the following planes which contain no vectors:
   a) the plane that contains the points $(1,0,1)$, $(2,2,4)$, and $(4,2,2)$.
      $-4(x-1)+8(y)-4(z-1)=0$
   b) the tangent plane to the graph of $f(x,y)=x^2+xy+y^2$ at the point $(1,2,7)$.
      $z=f(1,2)+f_x(1,2)(x-1)+f_y(1,2)(y-2)=7+4(x-1)+5(y-2)$
   c) the tangent plane to the surface defined by the equation $x^2+y^4+z^6=3$ at the point $(1,1,1)$
      $2(x-1)+4(y-1)+6(z-1)=0$
3) Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined $f(x,y)=2x-x^2-y^2$, and let $B=\left\{(x,y)\,\middle|\,x^2+y^2\le 4\right\}$
   a) Find the critical point(s) of f.
      (1,0)
   b) Using the 2nd Derivative test, determine the nature of f at the critical point(s), i.e. a relative maximum, relative minimum, or a saddle. Make sure to explain your answer completely.
      Relative maximum since $f_{xx}(1,2)<0$ and the discriminant of the Hessian is >0.

c) Find the absolute minimum and maximum of $f|_B$, that is f restricted to the set B.

Check critical point and the points satisfying

$$\begin{cases} 2-2x=\lambda(2x) \\ -2y=\lambda(2y) \\ x^2+y^2=4 \end{cases}$$

By the second equation, if $y\neq 0$ then $\lambda=-1$. Plugging this into the first equation, we get $2=0$ which is a contradiction, so this case never happens. On the other hand, if $y=0$, we find that $x=\pm 2$.

Checking these 3 points, we find the max to be $f(1,0)=1$ and the min to be $f(-2,0)=-12$

4) Suppose that $F(x,y,z)=x^2+y^4+x^2z^2$ gives the concentration of salt in a fluid at the point (x,y,z), and that you are at the point $(-1,1,1)$.

a) Find a unit vector in the direction that you should move if you want the concentration to increase the fastest.

$$\frac{\nabla F(-1,1,1)}{\|\nabla F(-1,1,1)\|}=\frac{-4\mathbf{i}+4\mathbf{j}+2\mathbf{k}}{\sqrt{4^2+4^2+2^2}}=\frac{-2}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{1}{3}\mathbf{k}$$

b) Suppose you start to move in the direction you found in part a) at a speed of 4 units per second. How fast is the concentration changing?

The norm of the gradient is the directional derivative in this direction, so the concentration is changing at a rate of $\|\nabla F(-1,1,1)\|4=24$ units of concentration per second.

p.75 (3) a) $L(t) = (-1,2,-1) + t(0,1,0)$

b) $(-3,1,0) - (0,2,-1) = (-3,-1,1)$, so $L(t) = (-3,1,0) + t(-3,-1,1)$

c) $-2(x+1) + (y-1) + 2(z-3) = 0$

(4) a) $(0,1,0) + t(3,0,1) = L(t)$

b) $(0,1,0) - (0,1,1) = (0,0,-1)$, so $L(t) = (0,1,0) + t(0,0,-1)$

c) $-1(x-1) + 1(y-1) - 1(z-1) = 0$

(5) a) $(-1,1,0)\cdot(0,0,1) = 0+0+0 = 0$

b) $(1,2,-1)\cdot(3,1,0) = 3+2+0 = 5$

c) $(-2,-1,1)\cdot(3,2,-2) = -6-2-2 = -10$

(6) a) $(-i+j)\times k = -(i\times k) + j\times k = -(-j) + i = \vec{i}+\vec{j}$

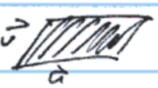
b) $\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & 1 & 0 \end{vmatrix} = \cancel{i(2+1) - j(1+0) + k(-1+2) = (1,1,1)}$

$i(\cancel{1+2}\ 0+1) - j(0+3) + k(1-6) = \cancel{1,-3,-5}\ (1,-3,-5)$

c) $\begin{vmatrix} i & j & k \\ -2 & -1 & 1 \\ 3 & 2 & -2 \end{vmatrix} = i(2-2) - j(4-3) + k(-4+3) = (0,-1,-1)$

(7) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$, so $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

a) $\cos\theta = 0$ b) $\cos\theta = \dfrac{5}{\sqrt{6}\cdot\sqrt{10}}$ c) $\cos\theta = \dfrac{-10}{\sqrt{6}\cdot\sqrt{17}}$

(8) ~~#~~ area of  $= \|\vec{u}\times\vec{v}\|$

a) $\sqrt{2}$ b) $\sqrt{1+9+25} = \sqrt{35}$ c) $\sqrt{2}$

pp. 167–8 30 a) $\|\vec{u}\| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\vec{u}\cdot\vec{v} = 2 - 2 - 6 = -6$

$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 2 & 1 & -3 \end{vmatrix} = i(6-2) - j(-3-4) + k(1+4) = (4,7,5)$

unit vector in direction of $\vec{u}$ is $\frac{\vec{u}}{\|\vec{u}\|} = \frac{(1,-2,2)}{3} = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}\right)$

b) $f(x,y,z) = e^{xy}\sin(xyz)$. We want

$D_{\vec{u}} f(0,1,1) = \nabla f(0,1,1)\cdot\frac{\vec{u}}{\|\vec{u}\|}$ (not $\nabla f(0,1,1)\cdot\vec{u}$ ← not a unit vector)

$\nabla f(x,y,z) = \left(ye^{xy}\sin(xyz) + e^{xy}yz\cos(xyz),\ xe^{xy}\sin(xyz) + e^{xy}xz\cos(xyz),\ e^{xy}xy\cos(xyz)\right)$

$\nabla f(0,1,1) = (1,0,0)$

$(1,0,0)\cdot\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \boxed{\frac{1}{3}}$

40 f = temp

$f_y = -0.2$, $f_x = -0.3$, so $\nabla f = (-0.3, -0.2)$

NE is the direction given by $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

So we want $20\cdot(-0.3,-0.2)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{-1}{2\sqrt{2}}\cdot 20 = \frac{-10}{\sqrt{2}}$

p 245 23 $Q(x,y,z) = xyz = 1000$ ← constraint

$P(x,y,z) = \text{Price} = 6x + 4y + 8z$ ← minimize this

– $\nabla Q(x,y,z) = (yz, xz, xy)$ – x, y, or z has to be 0 for this to be $\vec{0}$, but those pts. don't satisfy $xyz = 1000$

– $\nabla P(x,y,z) = (6,4,8) = \lambda(yz, xz, xy)$

$x\cdot 6 = \lambda yz\cdot x$

$y\cdot 4 = \lambda xz\cdot y$ so $6x = 4y = 8z$, so $x = \frac{4}{3}z$, $y = 2z$

$z\cdot 8 = \lambda xy\cdot z$

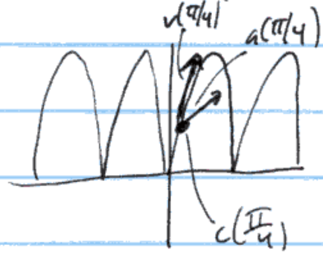
$xyz = 1000$, $\frac{4}{3}z\cdot 2z\cdot z = 1000$, $\frac{8}{3}z^3 = 1000$, $z^3 = 375$, so

$x = \frac{20}{3}\sqrt[3]{3}$, $y = 10\sqrt[3]{3}$, $z = 5\sqrt[3]{3}$

p290 6) $c(t) = (t - \sin t,\ 1 - \cos t)$

$v(t) = c'(t) = (1 - \cos t,\ \sin t)$ $\quad v(\frac{\pi}{4}) = (1 - \frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2})$

$a(t) = c''(t) = (\sin t,\ \cos t)$ $\quad a(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2})$



7) $F = ma$ $\quad c(t) = (t^2, \sin t, \cos t)$

$v(t) = c'(t) = (2t, \cos t, -\sin t)$

$a(t) = c''(t) = (2, -\sin t, -\cos t)$

So $F(0) = ma(0) = m(2, 0, -1) = (2m, 0, -m)$

8) a) $\|c(t)\|$ is constant, so $\|c(t)\|^2 = c(t) \cdot c(t)$ is constant.

Thus $\frac{d}{dt}(c(t) \cdot c(t)) = 0$

$c(t) \cdot c'(t) + c'(t) \cdot c(t) = 0$

$2\, c(t) \cdot c'(t) = 0$, so $c(t) \cdot c'(t) = 0$, ie, $c(t) \perp c'(t)$

b) constant speed means $\|c'(t)\|$ is constant, so $\|c'(t)\|^2$ is also constant, so $\frac{d}{dt}(c'(t) \cdot c'(t)) = 0$

$c'(t) \cdot c''(t) + c''(t) c'(t) = 0$

$2\, c'(t) c''(t) = 0$, so $c'(t) \cdot c''(t) = 0$, ie,

$c'(t) \perp c''(t)$.