Math 18, Solns. to selected hw problems, §§7.3–7.6, due 11/21

7.3.12, 7.5.7, & 7.6.2

(7.3.12) The plane is given by the eqn. $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$, where $\vec{n}=(a,b,c)$ is a normal vector to the surface and $(x_0,y_0,z_0)=(1,1,\sqrt{2})$. So we just need to find the normal vector $\vec{n}$.

a) $\vec{n}=T_\theta \times T_\varphi$. $T_\theta=(-2\sin\theta\sin\varphi, 2\cos\theta\sin\varphi, 0)$

$T_\varphi=(2\cos\theta\cos\varphi, 2\sin\theta\cos\varphi, -2\sin\varphi)$

and $T_\theta\times T_\varphi = 4(-\cos\theta\sin^2\varphi, -\sin\theta\sin^2\varphi, -\cos\varphi\sin\varphi)$

Since $z=2\cos\varphi=\sqrt{2}$, $\cos\varphi=\frac{1}{\sqrt{2}}$, so $\sin\varphi=\frac{1}{\sqrt{2}}$ (since $0\le\varphi\le\pi$). Plugging this in with $x=2\cos\theta\sin\varphi=1$ & $y=2\sin\theta\sin\varphi=1$, we get $\vec{n}=T_\theta\times T_\varphi=(-\sqrt{2},-\sqrt{2},-2)$

b) A normal vector to a level surface is the gradient ∇f, so take $\vec{n}=(2x,2y,2z)=\nabla f$. At $(1,1,\sqrt{2})$, this is $\vec{n}=(2,2,2\sqrt{2})$. This is a constant multiple $(-\sqrt{2})$ of the $\vec{n}$ from part a, as it must be.

c) Tangent plane to a graph: $z=z_0+g_x(x-x_0)+g_y(y-y_0)$, or $\vec{n}=(g_x,g_y,-1)$

$g_x=\frac{-x}{\sqrt{4-x^2-y^2}}$ & $g_y=\frac{-y}{\sqrt{4-x^2-y^2}}$. At $x=1, y=1$, this gives $g_x=\frac{-1}{\sqrt{2}}=g_y$.

So $\vec{n}=(\frac{-1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-1)$, which again is a constant multiple of the vectors from a & b.

(7.5.7) First parametrize, say $x=r\cos\theta$, $y=r\sin\theta$, $z=r^2$, with $0\le r\le 1$, $0\le\theta\le 2\pi$

Then $\iint_S z\,dS=\int_0^{2\pi}\int_0^1 r^2\|T_r\times T_\theta\|\,dr\,d\theta$. $T_r=(\cos\theta,\sin\theta,2r)$ &

$T_\theta=(-r\sin\theta, r\cos\theta, 0)$, so $T_r\times T_\theta=(-2r^2\cos\theta, -2r^2\sin\theta, r)$, and

$\|T_r\times T_\theta\|=r\sqrt{4r^2+1}$. So we solve $\int_0^{2\pi}\int_0^1 r^3\sqrt{4r^2+1}\,dr\,d\theta = 2\left(\frac{1}{120}+\frac{5\sqrt{5}}{24}\right)\pi$

(7.6.2) The vector field is $\vec{F}=-\nabla T=(-1,0,0)$. By symmetry, the heat flux across the sphere should be 0 — the heat flowing in is equal to the heat flowing out. To compute, parametrize S: $x=\sin\varphi\cos\theta$, $y=\sin\varphi\sin\theta$, and $z=\cos\varphi$, $0\le\theta\le 2\pi$, $0\le\varphi\le\pi$. Then the heat flux is

$\iint_S -\nabla T\cdot d\vec{S}=\int_0^{2\pi}\int_0^\pi(-1,0,0)\cdot(T_\theta\times T_\varphi)\,d\varphi\,d\theta=\int_0^{2\pi}\int_0^\pi \sin^2\varphi\cos\theta\,d\varphi\,d\theta$

$=0$.