Math 18 Solutions to selected HW problems, §§4.3 & 4.4, due 10/24

4.3: #17 4.4: # 4, 14, 36

4.3.17 a) $E(t) = \frac{1}{2} m\, r'(t)\cdot r'(t) + V(r(t))$

$$\frac{dE}{dt} = \frac{1}{2} m\left(r''(t)\cdot r'(t) + r'(t)\cdot r''(t)\right) + \nabla V(r(t)) \cdot (r'(t))$$

$$= \underline{m\, r''(t)\cdot r'(t) + \nabla V(r(t))\cdot r'(t)}$$

Since $F = ma = -\nabla V$ & $a = r''(t)$, this equals 0. $\frac{dE}{dt} = 0$, so E is constant.

b) $V(r(t))$ is constant (that's what it means to move on an equipotential surface), so $\frac{1}{2} m \|r'(t)\|^2 = E - V(r(t))$ is constant, so the square of the speed $\|r'(t)\|$ is constant, so the speed is constant.

4.4.4 $V(x,y,z) = \left(x^2, (x+y)^2, (x+y+z)^2\right)$

$$\nabla\cdot V = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}\left((x+y)^2\right) + \frac{\partial}{\partial z}\left((x+y+z)^2\right)$$

$$= 2x + 2(x+y) + 2(x+y+z) = 6x + 4y + 2z$$

4.4.14 $F(x,y,z) = (yz, xz, xy)$

$$\text{curl } F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = i(x-x) - j(y-y) + k(z-z) = (0,0,0)$$

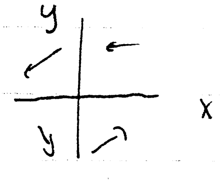
(This makes sense because $F = \nabla f$, where $f(x,y,z) = xyz$, and $\nabla\times(\nabla f) = \vec{0}$)

4.4.36 $V(x,y,z) = (-y, x, 0)$

a) $\text{div } V = \nabla\cdot V = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0+0+0 = 0$

$$\text{curl } V = \nabla\times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = i(0-0) - j(0-0) + k(1-(-1)) = 2k$$

$$= (0,0,2)$$

b) 

Flow lines ~~are~~ are $(C\cos t, C\sin t, D)$ (C & D are constants)

c) The paddle wheel will turn with its axis pointing in the direction of the curl, ie, in the $\vec{k}$ direction.