Math 18 Practice Midterm 2 Solns

1. Arc length is $\int ds = \int_1^2 \|c'(t)\|\,dt = \int_1^2 \|(1, \frac{1}{t}, 0)\|\,dt$

$$= \int_1^2 \sqrt{1^2 + \left(\tfrac{1}{t}\right)^2 + 0^2}\,dt = \int_1^2 \sqrt{1 + \tfrac{1}{t^2}}\,dt = -\sqrt{2} + \sqrt{5} + \ln(2) + \ln(1+\sqrt{2}) - \ln(1+\sqrt{5})$$

2. Need to show: $c'(t) = F(c(t))$

$$c'(t) = \left(\left(\tfrac{1}{1-t}\right)', 0', \left(\tfrac{e^t}{1-t}\right)'\right) = \left(\tfrac{1}{(1-t)^2}, 0, \tfrac{e^t}{1-t}\left(1 + \tfrac{1}{1-t}\right)\right)$$

& $F(c(t)) = \left(\left(\frac{1}{1-t}\right)^2, 0, \frac{e^t}{1-t}\left(1 + \frac{1}{1-t}\right)\right)$ ✓

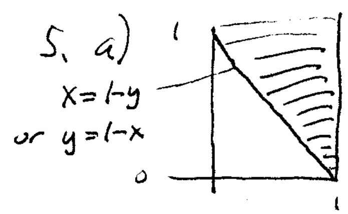
3. a) $\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (2x, 3y, 4z) = 2 + 3 + 4 = 9$

$$\nabla \times F = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3y & 4z \end{pmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = (0,0,0)$$

b) Similarly, $\nabla \cdot F = 1 + 1 + 1 = 3$, & $\nabla \times F = (-1, -1, -1)$

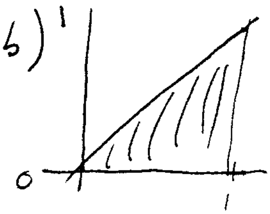
4. If we think of the vector field as representing the flow of a gas, then the divergence measures how much the gas is expanding. So the divergence in (a) is greater.

5. a)



$$\int_0^1 \int_{1-x}^1 (x + y^2)\,dy\,dx = \frac{7}{12}$$

b)



$$\int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \frac{1}{2}(1 - \cos 1)$$

6. $\int_0^1 \int_0^x \int_0^y (y + xz)\,dz\,dy\,dx = \int_0^1 \int_0^x \left(y^2 + \frac{1}{2}xy^2\right)dy\,dx$

$$= \int_0^1 \left(\frac{1}{3}x^3 + \frac{1}{6}x^4\right)dx = \frac{1}{12} + \frac{1}{30} = \frac{5}{60} + \frac{2}{60} = \frac{7}{60}$$

7. a) The xy-region is the disk of radius 1 centered at (0,0), and $(x^2+y^2)^{\frac{1}{2}} = r$, so it's

$$\int_0^1 \int_0^1 \int_0^{2\pi} r \cdot r \, d\theta \, dr \, dz$$

b) This is the region above the cone $\varphi = \frac{\pi}{4}$ & below the sphere $\rho = 2$. (The "shadow" in the xy-plane is the disk of radius $\sqrt{2}$ centered at (0,0)) $z^2 = (\rho\cos\varphi)^2$, so it's

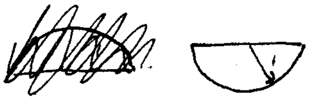
$$\int_0^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^2 \cos^2\varphi \, \rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho$$

8. Mass $= \iiint_R \delta \, dV = \int_1^2 \int_0^{2\pi} \int_0^{\pi} \left(\begin{smallmatrix} 10^6 \\ 0.4 \end{smallmatrix}\right)/\rho^2 \cdot \rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho = \underline{(9.92 \times 10^6)\pi}$ – units

9. $d^2 = x^2+y^2+z^2$, so the average is $\dfrac{\iiint_C 32(x^2+y^2+z^2)\,dV}{\iiint_C dV}$

$$= \frac{\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 32(x^2+y^2+z^2)\,dx\,dy\,dz}{2^3} = \underline{32}$$

10. Change to polar: $-x^2-y^2 = -r^2$,



$$\int_0^1 \int_{\pi}^{2\pi} e^{-r^2} r \, d\theta \, dr = \left(\frac{1}{2} - \frac{1}{2e}\right)\pi$$

11. $\int_C f \, ds = \int_0^1 f(c(t)) \, \|c'(t)\| \, dt = \int_0^1 \frac{3}{2}t^2 \cdot 2t^2 \cdot t \, \|(3t, 4t, 1)\| \, dt$

$$= \int_0^1 3t^5 \sqrt{25t^2+1} \, dt = \frac{236{,}158\sqrt{26} - 8}{35 \cdot 25^3}$$

12. $\int_0^{7\pi/2} \left(\sin\theta \cdot -3\cos^2\theta \sin\theta + \cos\theta \cdot 3\sin^2\theta\cos\theta - (\cos^3\theta \sin^3\theta)^{1/3}\right) d\theta$

$$= \int_0^{7\pi/2} -\cos\theta\sin\theta = \frac{1}{2}\cos^2\theta \Big|_0^{7\pi/2} = 0 - \frac{1}{2} = \frac{-1}{2}$$

13. It's 0 on C_1 & C_2, because the parts going in the same direction as F are cancelled by those in the opposite direction. (It's positive on C_3).