1. (15 points) Compute the integral of the function $f(x,y,z) = z$ over the curve $\mathbf{c}(t) = (\cos t, \sin t, -t), 0 \le t \le 2$.

$$\int_c f\,ds = \int_0^2 f(c(t))\,\|c'(t)\|\,dt = \int_0^2 (-t)\,\|(-\sin t, \cos t, -1)\|\,dt$$

$$= \int_0^2 (-t)\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \int_0^2 (-t)\sqrt{2} = -\frac{\sqrt{2}}{2}t^2\Big|_0^2 = -2\sqrt{2}$$

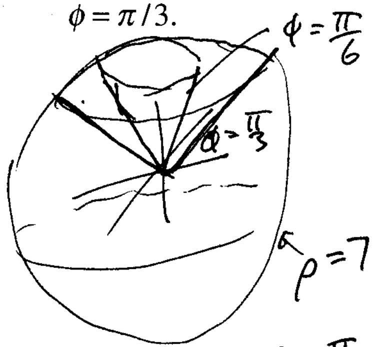
2. (14 points) Calculate the divergence and the curl of the vector field $\mathbf{F}(x,y,z) = (-x^2 + y^2, y + z, -z + x)$.

$$\operatorname{div} F = \nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(-x^2+y^2, y+z, -z+x\right) = -2x + 1 - 1 = -2x$$

$$\operatorname{curl} F = \nabla \times F = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x^2+y^2 & y+z & -z+x \end{pmatrix} = \vec{i}\,(\overset{0+1}{\cancel{0}}) - \vec{j}\,(1-0) + \vec{k}\,(0-2y)$$

$$= (1, -1, -2y)$$

3. (15 points) Set up, but **do not evaluate**, an integral, in spherical coordinates, for the volume of the region in the **first octant** of $\mathbf{R}^3$ which is bounded by the sphere $\rho = 7$ and the cones $\phi = \pi/6$ and $\phi = \pi/3$.



1st octant: $0 \le \theta \le \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_0^{7} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

4. (15 points) Compute the area of the elliptical disk $D = \{(x,y) \mid 4x^2 + 9y^2 \le 36\}$. **HINT:** The change of variables $T(u,v) = (3u, 2v)$ maps the unit disk $D^* = \{(u,v) \mid u^2 + v^2 \le 1\}$ one-to-one onto D.

$$\text{Area of } D = \iint_D dx\,dy = \iint_{D^*} |\det DT| \, du\,dv$$

$$|\det DT| = \left|\det\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\right| = 6, \text{ so } = \iint_{D^*} 6 \, du\,dv$$

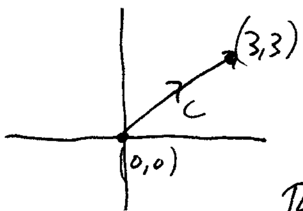
$$= 6 \iint_{D^*} du\,dv$$

$$= 6(\text{Area } D^*)$$

$$= 6\pi$$

5. (15 points) How much work is done by the force $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$ on a particle that moves in a straight line from the point (0,0) to the point (3,3)?

$$\text{Work} = \int_C \vec{F} \cdot d\vec{s}$$



parameterize C, say by $\vec{c}(t) = (t,t), \quad 0 \le t \le 3$

Then $$\int_C \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F}(\vec{c}(t)) \cdot \vec{c}\,'(t)\,dt$$

$$= \int_0^3 (t,t) \cdot (1,1)\,dt$$

$$= \int_0^3 (t+t)\,dt = \int_0^3 2t\,dt = t^2 \Big|_0^3 = 9$$

— OR —

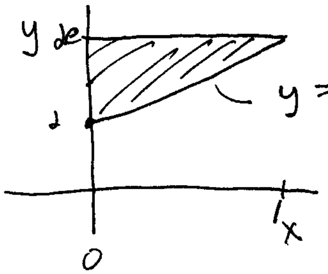
observe that $F(x,y) = (x,y)$ is the gradient ∇f of the function $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$. Thus $\int_C \vec{F} \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s}$

$$= f(3,3) - f(0,0)$$

$$= \frac{9}{2} + \frac{9}{2} - \left(\frac{0}{2} + \frac{0}{2}\right)$$

$$= 9$$

6. (10 points) Change the order of integration of $\int_0^1 \int_{2e^x}^{2e} f(x,y)\, dy\, dx$, but **do not evaluate**.



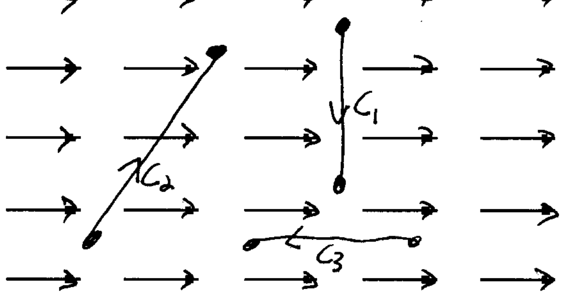
$y = 2e^x$,

or $x = \ln \frac{y}{2}$

$= \ln y - \ln 2$

So it's $\int_2^{2e} \int_0^{\ln \frac{y}{2}} f(x,y)\, dx\, dy$

7. (8 points) Consider the constant vector field $\mathbf{F}(x,y) = \mathbf{i}$ shown in the figure below, together with the paths $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$. Arrange the line integrals $\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s}$, $\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$, and $\int_{\mathbf{c}_3} \mathbf{F} \cdot d\mathbf{s}$ in order from least to greatest.



c_1 is $\perp$ to the v.f., so $\int_{c_1} \vec{F} \cdot d\vec{s} = 0$

c_2 is roughly in the same direction as the v.f., so $\int_{c_2} \vec{F} \cdot d\vec{s} > 0$.

c_3 is directly against the v.f., so $\int_{c_3} \vec{F} \cdot d\vec{s} < 0$.

Thus $\int_{c_3} \vec{F} \cdot d\vec{s} < \int_{c_1} \vec{F} \cdot d\vec{s} < \int_{c_2} \vec{F} \cdot d\vec{s}$

8. (8 points) For each of the following statements, say whether it's true or false; if it's false, explain why.

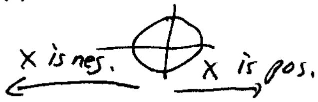
   a. If $f(x,y)=3$ for all points (x,y) in a region R, then $\int\int_R f(x,y)\, dA = 3 \cdot \text{Area}(R)$.

   True

   b. If $\iint_R f(x,y)\, dA = 0$, then $f(x,y)=0$ at every point (x,y) in R.

   False - pos. & neg. values of f can cancel out.

   Ex If R is the unit disk, $\iint_R x \, dx dy = 0$



   c. Let $\delta(x,y)$ be the population density of a city, in people per km^2. If R is a region in the city, then $\iint_R \delta(x,y)\, dA$ gives the total number of people in the region R.

   True

   d. The vector field $\mathbf{F}(x,y) = y\mathbf{i} + \mathbf{j}$ is a gradient vector field, i.e., $\mathbf{F} = \nabla f$ for some C^2 function $f : \mathbf{R}^2 \to \mathbf{R}$.

   False. Assume $(y,1) = \nabla f = (f_x, f_y)$ for some f.

   Then $f_{xy} = (f_x)_y = (y)_y = 1$, and

   $f_{yx} = (f_y)_x = (1)_x = 0$.

   But $f_{xy} = f_{yx}$ for a C^2 function, so this is impossible.

EXTRA CREDIT (2 points) True or false: The following is funny. Explain.

Q: What do you get when you cross an elephant and a banana?

A: $\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ & \text{elephant} & \\ & \text{banana} & \end{pmatrix}$