(7) (15 points) Use Stokes' Theorem to compute $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $S$ is the ellipsoid $x^2 + 2y^2 + 3z^2 = 10$ and $\mathbf{F}$ is the vector field

$$\mathbf{F}(x, y, z) = (\cos(\ln(xz)),\ (x^2 + y^2)^{\frac{5}{2}},\ e^{z^2 - x}).$$

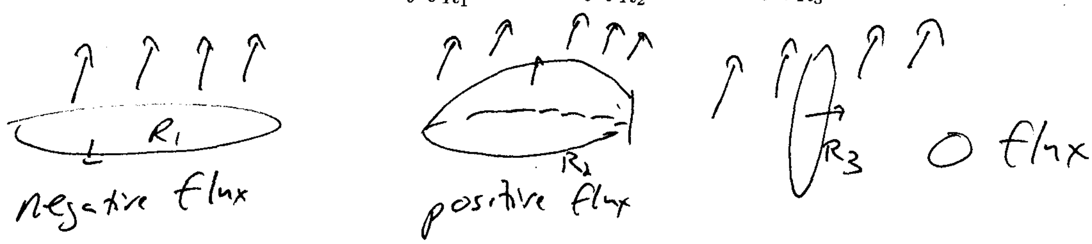
Stokes' Th. says that $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{s}$, where $C$ is the boundary curve of $S$. Since $S$ has no boundary curve, $C$ is empty, and the integral is $0$.

(8) (15 points) Let $\mathbf{F}(x, y, z) = \mathbf{k}$. Define the following surfaces in $\mathbb{R}^3$:

- $R_1$ is the unit disk in the $xy$-plane, oriented in the negative $z$ direction.
- $R_2$ is the upper hemisphere of the unit sphere, i.e., $R_2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$, oriented in the positive $z$ direction.
- $R_3$ is the unit disk in the $yz$-plane, oriented in the positive $x$ direction.

Place the following flux integrals in order from least to greatest:

$$\iint_{R_1} \mathbf{F} \cdot d\mathbf{S},\quad \iint_{R_2} \mathbf{F} \cdot d\mathbf{S},\quad \iint_{R_3} \mathbf{F} \cdot d\mathbf{S}$$




so: $\iint_{R_1} \vec{F} \cdot d\vec{s} < \iint_{R_3} \vec{F} \cdot d\vec{s} < \iint_{R_2} \vec{F} \cdot d\vec{s}$

(9) (20 points) Let S be the surface parametrized by

$$x = 1 + \cos u, \quad y = 4 + 3\sin u, \quad z = v, \quad 0 \le u \le 2\pi, \quad -2 \le v \le 2.$$

Find an equation for the tangent plane to the surface $S$ at the point corresponding to $u = 0,\ v = 1$.

Plane is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$,

where $(x_0, y_0, z_0)$ is a pt. in the plane & $(a, b, c)$ is a normal vector.

$(x_0, y_0, z_0) = (1 + \cos 0,\ 4 + 3\sin 0,\ 1) = (2, 4, 1)$

A normal vector is $T_u \times T_v$.

$T_u = (-\sin u,\ 3\cos u,\ 0)$

$T_v = (0,\ 0,\ 1)$

So $T_u \times T_v = \det \begin{pmatrix} i & j & k \\ -\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} = \vec{i}(3\cos u) - \vec{j}(-\sin u) + \vec{k}(0)$

$= (3\cos u,\ \sin u,\ 0)$

At $u = 0,\ v = 1$, this is $(3, 0, 0)$.

So the plane is $3(x-2) + 0(y-4) + 0(z-1) = 0$

or $3x = 6$

or $x = 2$

(10) (20 points) Let $D$ be the elliptical disk $25x^2 + 4y^2 \leq 100$. Compute the area of $D$ in two different ways:

(a) Using the fact that the change of variables $T(u,v) = (2u, 5v)$ maps the unit disk $D^*$ in the $(u,v)$-plane one-to-one onto $D$.

area $(D) = \iint_D dx\,dy = \iint_{D^*} |\det DT|\,du\,dv$ (change of variables Theorem)

$DT = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$, so $|\det DT| = 10$.

Thus area $(D) = \iint_{D^*} 10\,du\,dv = 10(\text{area } D^*) = 10 \cdot \pi$

(b) Using Green's Theorem. (HINT: A parametrization for the ellipse $25x^2 + 4y^2 = 100$ is $x = 2\cos\theta$, $y = 5\sin\theta$, $0 \leq \theta \leq 2\pi$.)

Let $C$ be the boundary ellipse. Then, by the corollary to Green's Theorem, area $(D) = \frac{1}{2}\int_C -y\,dx + x\,dy$

$(x,y) = (2\cos\theta, 5\sin\theta)$ $dx = -2\sin\theta\,d\theta$ $dy = 5\cos\theta\,d\theta$

So $\frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi}\left(-5\sin\theta(-2\sin\theta) + 2\cos\theta(5\cos\theta)\right)d\theta$

$= \frac{1}{2}\int_0^{2\pi}(10\sin^2\theta + 10\cos^2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{2\pi} 10\,d\theta = \frac{1}{2}\cdot 20\pi = 10\pi$

(11) (20 points) Define the functions

$$g(x,y) = (g_1(x,y), g_2(x,y)) = (x^2+1, y^2),$$
$$f(u,v) = (f_1(u,v), f_2(u,v), f_3(u,v)) = (u+v, u, v^2), \text{ and}$$
$$h(x,y) = (h_1(x,y), h_2(x,y), h_3(x,y)) = f(g(x,y)).$$

(a) Compute $\mathbf{D}h(1,1)$.

$\underline{Dh(1,1) = Df(g(1,1)) \cdot Dg(1,1)}$ $\quad g(1,1) = (2,1)$

$Df(u,v) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$, so $Df(g(1,1)) = Df(2,1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$

$Dg(x,y) = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$, so $Dg(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Thus $Dh(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$

(b) What is $\dfrac{\partial h_3}{\partial y}(1,1)$?

$Dh(1,1) = \begin{pmatrix} \frac{\partial h_1}{\partial x}(1,1) & \frac{\partial h_1}{\partial y}(1,1) \\ \frac{\partial h_2}{\partial x}(1,1) & \frac{\partial h_2}{\partial y}(1,1) \\ \frac{\partial h_3}{\partial x}(1,1) & \frac{\partial h_3}{\partial y}(1,1) \end{pmatrix}$ by definition, so $\frac{\partial h_3}{\partial y}(1,1)$ is the lower right entry of $Dh(1,1)$ is $4$.

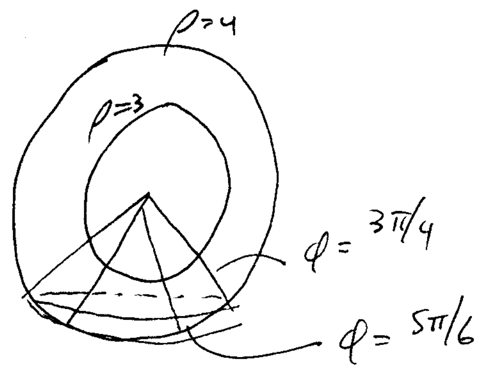
(12) (20 points) Let $R$ be the region in $\mathbb{R}^3$ bounded by the sphere of radius 3, the sphere of radius 4, the cone $z = -\sqrt{x^2+y^2}$, and the cone $z = -2\sqrt{x^2+y^2}$. Which of the following integrals represents the volume of $R$?

(a) $\displaystyle\int_3^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{x^2+y^2}}^{-2\sqrt{x^2+y^2}} dz\,dy\,dx$

(b) $\displaystyle\int_3^4 \int_0^{2\pi} \int_{\frac{3\pi}{4}}^{\frac{5\pi}{6}} d\phi\,d\theta\,d\rho$

(c) $\displaystyle\int_3^4 \int_0^{2\pi} \int_{-2r^2}^{-r^2} r\,dz\,d\theta\,dr$

(d) None of the above.




None of the above. It's

$$\int_3^4 \int_0^{2\pi} \int_{3\pi/4}^{5\pi/6} \rho^2 \sin\varphi \, d\varphi\, d\theta\, d\rho$$

EXTRA CREDIT (5 points) Which is your favorite: Gauss's, Stokes', or Green's Theorem? Why?

Oh, don't make me choose ---