(1) (15 pts) Consider the initial value problem $x' = y/x$, $y' = 2xy$, $x(0) = 1$, $y(0) = -4$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$ and $y(1)$.

Euler's method: $\vec{X}_n = \overrightarrow{X_{n-1}} + h \cdot \vec{V}(\vec{X}_{n-1})$, where $\vec{V}$ is the vector field. Here, $\vec{V}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/x \\ 2xy \end{pmatrix}$ and $h = \frac{1}{2}$.

$$\begin{pmatrix} x(\frac{1}{2}) \\ y(\frac{1}{2}) \end{pmatrix} \approx \vec{X}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -4 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} \approx \vec{X}_2 = \begin{pmatrix} -1 \\ -8 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 8 \\ 16 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

So $x(1) \approx 3$ and $y(1) \approx 0$

This isn't likely to be correct, since for x to get from 1 to -1, it would have to pass through 0, and $x' = \frac{y}{x}$ isn't defined when $x = 0$

(2) (8 pts) Let $x(t)$ be Country 1's annual expenditure on arms, and $y(t)$ Country 2's. One of the systems below models the arms race between two countries that have historically gotten along pretty well, and the other system models the race between two countries that have fought a lot in the past. Which is which, and how do you know?

a) $\begin{aligned} x' &= 4y - x - 2 \\ y' &= 3x - 2y - 3 \end{aligned}$ b) $\begin{aligned} x' &= 3y - x + 1 \\ y' &= 4x - 2y + 2 \end{aligned}$

a) is the one where they've gotten along, b) the one where they've fought. You can tell by the sign of the constant terms. Negative terms hinder growth (they don't want to fight each other), while positive terms encourage growth (they do want to fight).

(3) Let $x(t)$ and $y(t)$ be the populations of zebras and horses, two species that complete with each other for delicious grass. A model for their behavior is

$$\begin{aligned} x' &= 14x - 2x^2 - xy \\ y' &= 16y - 2y^2 - xy \end{aligned}.$$

(a) Find the four equilibrium points for the system. Do they all make physical sense?

Solve $x' = y' = 0$.

$14x - 2x^2 - xy = 0$ $16y - 2y^2 - xy = 0$

$x(14 - 2x - y) = 0$ $y(16 - 2y - x) = 0$

$x = 0$ or $y = 14 - 2x$ $y = 0$ or $y = 8 - \frac{x}{2}$

$14 - 2x = 8 - \frac{x}{2}$

$6 = \frac{3}{2}x$

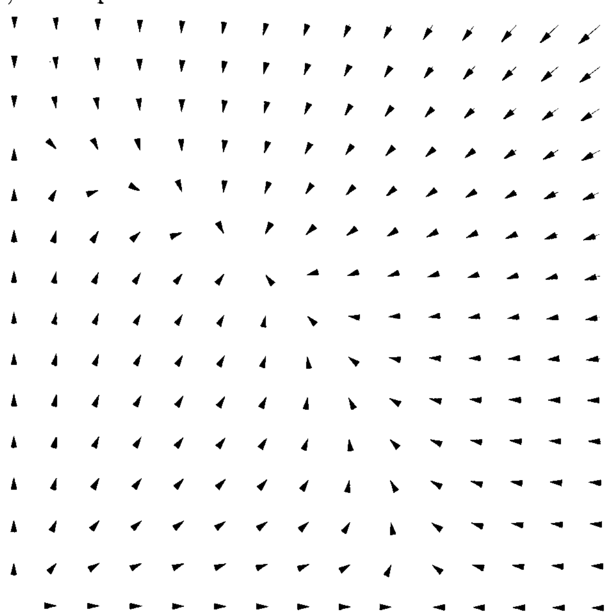
$4 = x$, $y = 6$

Solns. are $(0,0)$, $(0,8)$, $(4,6)$, $(7,0)$ — They all make sense, since there are no negative animals.

(b) The figure below shows the vector field for the system (the lower left corner is the origin of the xy-plane). What are the possible long-term behaviors of solutions?

Start with no animals, end with no animals. Start with some zebras & no horses, head toward 7000 zebras & no horses. Start with some horses & ~~some~~ no zebras head toward 8000 horses. Otherwise, if you start with some of both, head towards 4000 zebras & 6000 horses.

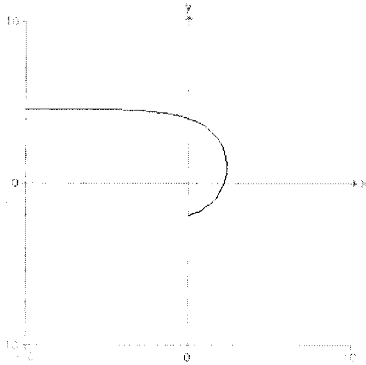
(c) Is it possible for zebras and horses to live together in loving harmony? Explain.



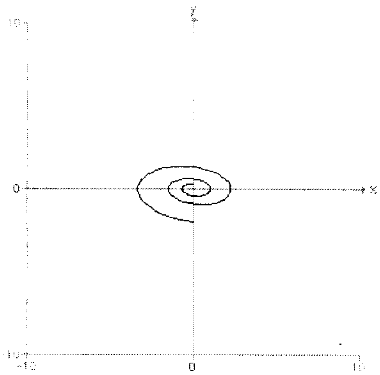
Yes - $(4,6)$ is a stable, attracting equilibrium.

(4) For each of the following three systems, a solution is pictured in the phase plane and as separate $x(t)$ and $y(t)$ graphs. Identify which phase plane solution and which set of x and y graphs goes with each system.

System	Phase Plane Solution	x and y Graphs
$x' = 1 - y$ $y' = e^{x/2}$	A	III
$x' = -y^2$ $y' = 3\cos x$	C	II
$x' = 2y$ $y' = -x/2 - y/4$	B	I



A.



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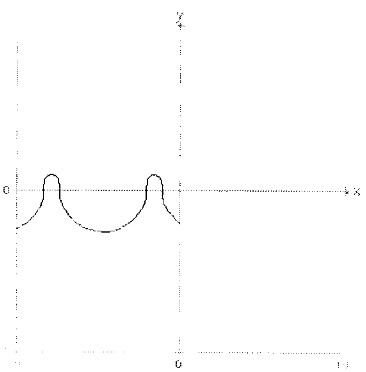

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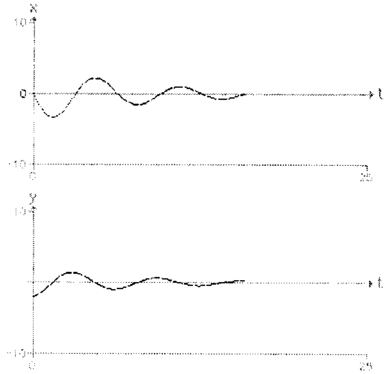
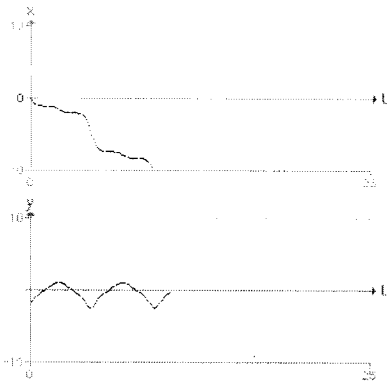
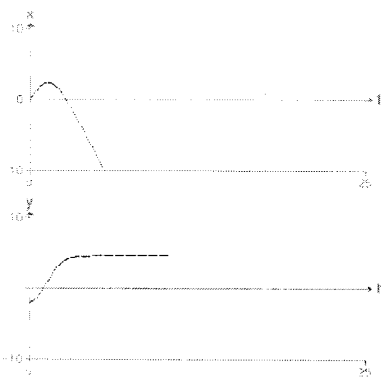


I.



II.



III.

(5) Monkey Town initially has 1500 residents: 600 green monkeys and 900 purple monkeys. Suppose that 100 green monkeys and 100 purple monkeys move in each month. Assume further that each month 100 monkeys move out, with the number of each color moving out proportional to their share of the total population of Monkey Town. When the population reaches 3000, how many ~~green~~ purple monkeys live in Monkey Town?

$\cancel{g' = 100 - 100\frac{g}{g+p}}$ $p = \#$ of purple monkeys

$$p' = \underset{\text{moving in}}{100} - \underset{\text{moving out}}{100 \cdot \frac{p}{\text{total monkeys}}} = 100 - 100 \cdot \frac{p}{\underset{\text{total pop. increases by 100 each month}}{1500 + 100t}}$$

So $p' = 100 - \frac{100}{1500+100t} p$

$p' + \frac{100}{1500+100t} p = 100$. Int. factor: $\mu' = \frac{100}{1500+100t} \mu$

$$\mu = e^{\ln(1500+100t)} \quad \left(\begin{array}{c}1500+100t > 0 \\ \text{always} \\ (t \geq 0)\end{array}\right)$$

$$\mu = 1500 + 100t$$

$$\left((1500+100t)p\right)' = 150{,}000 + 10{,}000t$$

$$(1500+100t)p = 150{,}000t + 5000t^2 + C$$

$$p = \frac{1500t + 50t^2 + D}{15 + t}$$

Since $P(0) = 900$, $D = 15 \cdot 900 = 13{,}500$

The pop. is 3000 at $t = 15$, so we have

$$P(t) = \frac{1500(15) + 50(15)^2 + 15 \cdot 900}{2 \cdot 15} = \frac{1500 + 50 \cdot 15 + 900}{2}$$

$$= \cancel{\frac{9900}{2}} = \cancel{4950}\ \frac{3150}{2} = 1575.$$

(6) Consider the family of differential equations $y' = y^3 - ay$, where a is a parameter.

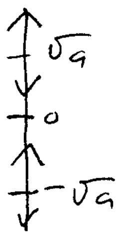
(a) What is the bifurcation value for a?

$y' = y(y^2 - a)$ - 3 roots if $a > 0$, 1 if $a \leq 0$.

So $a = 0$ is the bifurcation value.

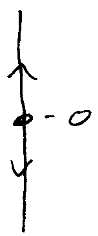
(b) Draw phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation value. In each case, what are the possible long-term behaviors?

$a > 0$



Pts. above $\sqrt{a}$ go to $+\infty$; pts. between $\sqrt{a}$ & $-\sqrt{a}$ go to 0, and pts. below $-\sqrt{a}$ go to $-\infty$.

$a = 0$



Pts. above 0 go to $+\infty$; pts. below to $-\infty$; 0 is fixed

$a < 0$

Same as $a = 0$.

(c) If $y(t)$ is the cockroach population in Sharples, what do you hope is true about a?

Hope that a is big & positive, & so the roaches will die off.

(7) Find the general solution to the differential equation $y' + y^2 \sin \cancel{t} = 0$.

$y' = -y^2 \sin t$

$\frac{1}{y^2}\,dy = -\sin t\,dt$ (if $y \neq 0$)

$-\frac{1}{y} = \cos t + C$

$y = \frac{-1}{\cos t + c}$ (exists as long as $\cos t \neq -c$)

or $y = 0$.