(1) (10 pts) Consider the initial value problem $y' = e^y$, $y(0) = 0$.

(a) Use Euler's method with step size $h = 0.5$ to approximate $y(1.5)$.

$$y(t+h) \approx y(t) + y'(t)\cdot h$$

So $y(\tfrac{1}{2}) \approx y(0) + y'(0)\cdot\tfrac{1}{2} = 0 + e^0\cdot\tfrac{1}{2} = \tfrac{1}{2}$

$y(1) \approx y(\tfrac{1}{2}) + y'(\tfrac{1}{2})\cdot\tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2}e^{1/2}$

$y(1.5) \approx y(1) + y'(1)\cdot\tfrac{1}{2} = \tfrac{1}{2} + \tfrac{1}{2}e^{1/2} + \tfrac{1}{2}e^{\frac{1}{2}+\frac{1}{2}e^{1/2}}$

(b) Solve the IVP analytically.

Separate variables: $e^{-y}\,dy = dt$

integrate $-e^{-y} = t + c$

$e^{-y} = d - t$

$-y = \ln(d-t)$: must have $d-t > 0$

$y = -\ln(d-t)$

Since $y(0) = 0 = -\ln(d-0)$, $d = 1$.

So $y = -\ln(1-t)$: exists only for $t < 1$

(c) How do you account for the discrepancy between your analytic solution from part (b) solution and your approximation in part (a)?

The real solution blows up at $t=1$. The slope y' is increasing too fast for Euler's method (which assumes that y' stays roughly constant) to give a reasonable approximation.

(2) Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} tx + ty + 1 - t^2 \\ 2tx - 2t^2 \end{pmatrix}. \qquad (*)$$

(a) Show that both $\mathbf{Y}_1(t) = \begin{pmatrix} t + e^{t^2} \\ e^{t^2} \end{pmatrix}$ and $\mathbf{Y}_2(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ are solutions of $(*)$.

$$Y_1'(t) = \begin{pmatrix} 1 + 2te^{t^2} \\ 2te^{t^2} \end{pmatrix} = \begin{pmatrix} t^2 + te^{t^2} + te^{t^2} + 1 - t^2 \\ 2t^2 + 2te^{t^2} - 2t^2 \end{pmatrix} = \begin{pmatrix} tx + ty + 1 - t^2 \\ 2tx - 2t^2 \end{pmatrix} ✓$$

$$Y_2'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} t^2 + t \cdot 0 + 1 - t^2 \\ 2t^2 - 2t^2 \end{pmatrix} = \begin{pmatrix} tx + ty + 1 - t^2 \\ 2tx - 2t^2 \end{pmatrix} ✓$$

(b) Show that $\mathbf{Y}_3(t) = e^{-t^2/2}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a solution of the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} tx + ty \\ 2tx \end{pmatrix}. \qquad (**)$$

$$Y_3'(t) = -t e^{-t^2/2}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = e^{-t^2/2}\begin{pmatrix} (-1)t + 2t \\ 2(-1)t \end{pmatrix} = \begin{pmatrix} tx + ty \\ 2tx \end{pmatrix} ✓$$

(c) Find the solution $\mathbf{Y}_0(t)$ of $(**)$ satisfying the initial condition $\mathbf{Y}_0(1) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

$Y_1 - Y_2 = e^{t^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a soln of $(**)$, the homogeneous eqn. associated to $(*)$. So $e^{t^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $e^{-t^2/2}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ form a basis for the solns of $(**)$. Plug in the initial condition, & get

$$Y_0(t) = 2e^{t^2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-t^2/2}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(3) Solve the initial value problem

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + 2ty \\ 2y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(HINT: The system partially decouples.)

First, solve for y: $y' = 2y$, so $y = Ce^{2t}$

Now, x: $x' = x + 2ty = x + 2Cte^{2t}$

$x' - x = 2Cte^{2t}$ integrating factor: $\mu(t) = e^{-t}$

$x'e^{-t} - xe^{-t} = 2Cte^{t}$

$(xe^{-t})' = 2Cte^{t}$ integrate both sides

$xe^{-t} = 2Cte^{t} - 2Ce^{t} + d$

$x = 2Cte^{2t} - 2Ce^{2t} + de^{t}$

$y(0) = 1 = Ce^{2\cdot 0} \Rightarrow C = 1$

$x(0) = 1 = 2\cdot 0 e^{2\cdot 0} - 2e^{2\cdot 0} + de^{0}$

$1 = -2 + d$

$3 = d$

So $x(t) = 2te^{2t} - 2e^{2t} + 3e^{t}$

$y(t) = e^{2t}$

(4) Let $x(t)$ and $y(t)$ be the populations (at time t) of two species of animals. The behavior of the two populations is modeled by the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2x + xy \\ -10y + 2xy \end{pmatrix}.$$

(a) What happens to each species in the absence of the other? How do you know?

If $y=0$, $x' = -2x$, so $x = Ce^{-2t}$: goes to 0 (dies out) as $t \to \infty$

If $x=0$, $y' = -10y$, so $y = de^{-10t}$: goes to 0 (dies out) as $t \to \infty$

(b) How do the two species get along? How do you know?

They get along well – they're good for each other

Coeff. of xy in x' is $1 > 0$ | interaction → increased growth
" " " in y' is $2 > 0$ | (xs) for both species

(c) What are the equilibrium population levels?

$x' = 0 = -2x + xy = x(-2+y)$ $x = 0$ or $y = 2$

$y' = 0 = -10y + 2xy = y(-10 + 2x)$. If $x=0$, $y=0$

If $y=2$, $x=5$

Equilibria: $(x=0, y=0)$ & $(x=5, y=2)$

(Continued on next page)

(4) (continued)

(d) Classify each equilibrium point. What does that mean, practically, about the population levels?

Jacobian is $\begin{pmatrix} -2+y & x \\ 2y & -10+2x \end{pmatrix}$

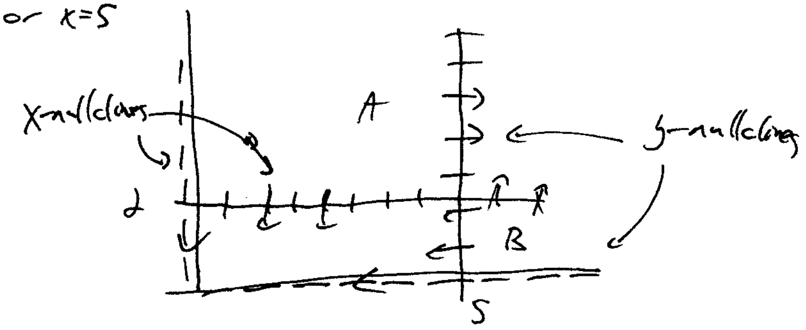
$Df(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -10 \end{pmatrix}$ $\lambda = -2, -10$: sink. If initial popns. are small, both species die out.

$Df(5,2) = \begin{pmatrix} 0 & 5 \\ 4 & 0 \end{pmatrix}$ $\lambda^2 - 20 = 0$, $\lambda = \pm\sqrt{20}$: saddle. This is an unstable equilibrium. If initial popns. are near this pt., they will almost certainly move away.

(e) Sketch the x- and y-nullclines of the system. (Be sure that I can tell which is which.)

x-nullcline : $x' = 0$: $x = 0$ or $y = 2$

y-nullcline : $y' = 0$: $y = 0$ or $x = 5$



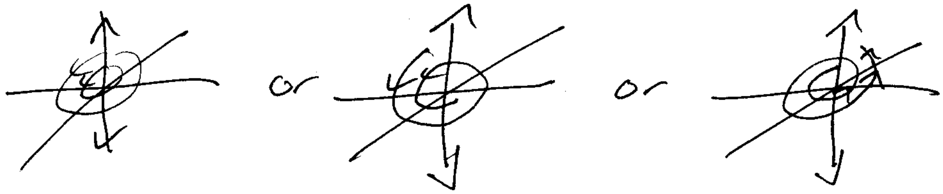
(f) Describe all the possible long-term behaviors of the system. Which of these behaviors might you actually see in nature? Why?

If x pop. starts below 5 & y pop. below 2, both will die off. Some other initial conditions also lead to extinction. If x starts above 5 and y above 2, both will grow without bound. A one-dimensional set of initial conditions is attracted to the saddle point $(5,2)$, but you won't observe this in nature, since it is unstable. If the initial popns are in A or B (above), they'll do one of these three things, but it's hard to say which.

(5) Consider the nonlinear system $\dfrac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. If $\mathbf{p_0} = (x_0, y_0, z_0)$ is an equilibrium point and $\mathbf{dF}(\mathbf{p_0})$ (the Jacobian matrix at $\mathbf{p_0}$) has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2i$, and $\lambda_3 = -2i$, what can you say about the behavior of the system near the point $\mathbf{p_0}$?

The linear system has one repelling direction ($\lambda_1 = 1$), and is a center in the other two dimensions.

The actual system will have one repelling direction for sure. In the other two dimensions, it could be a center, spiral source, or spiral sink.



(6) **(a)** Explain why the solutions of a linear homogeneous system of ordinary differential equations form a vector space.

Because: $\boxed{\vec{Y}' = A(t)\vec{Y}}$

if $\vec{Y}_1$ & $\vec{Y}_2$ are solns, so is any linear combination, $a\vec{Y}_1 + b\vec{Y}_2$

$$(a\vec{Y}_1 + b\vec{Y}_2)' = a\vec{Y}_1' + b\vec{Y}_2' = aA(t)\vec{Y}_1 + bA(t)\vec{Y}_2$$
$$= A(t)(a\vec{Y}_1 + b\vec{Y}_2)$$

(b) Give an example of a first-order ordinary differential equation whose solutions do *not* form a vector space. (Be sure to justify your answer, that is, show that they don't form a vector space.)

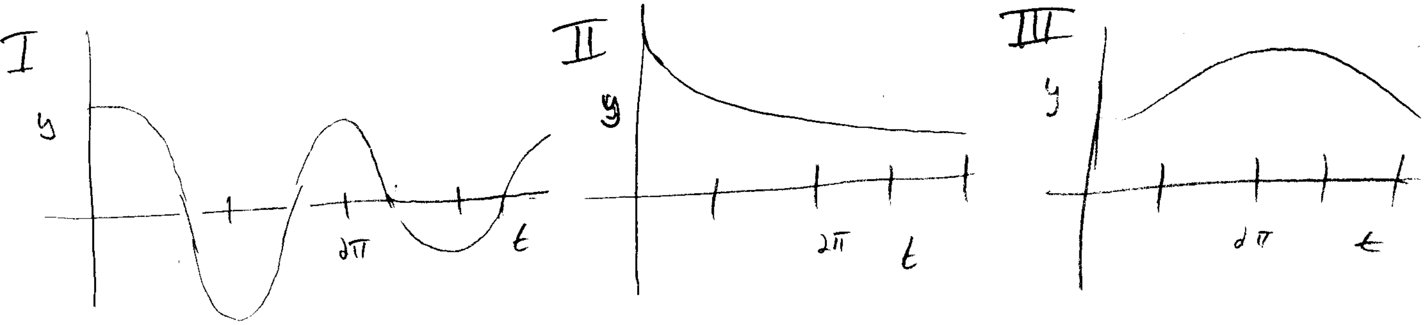
$y' = 1.$

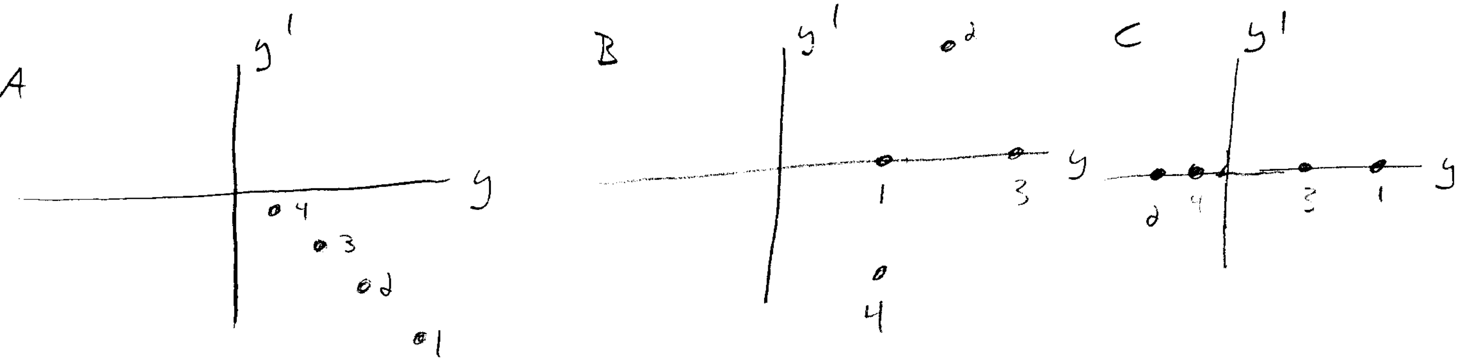
Then $y_1(t) = t$ & $y_2(t) = t + 1$ are solns, but $y_1(t) + y_2(t)$ is not, since

$$(y_1 + y_2)' = 2 \neq 1.$$

(7) In the figures below, Poincaré map pictures are given for orbits of three different systems, each with a periodic forcing term of period π. Also, three $y(t)$-graphs for solutions of the systems are given.

Match the Poincaré return map picture with the $y(t)$-graph. ~~Describe briefly the qualitative behavior of the solution.~~





I - C

II - A

III - B

(**8**) Find the general solution of the system

$$\mathbf{Y}' = \begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix} \mathbf{Y}.$$

Find eigenvalues/vectors: $\det\begin{pmatrix} -2-\lambda & 1 \\ -1 & -4-\lambda \end{pmatrix} = \lambda^2 + 6\lambda + 8 + 1 = \lambda^2 + 6\lambda + 9$

$(\lambda+3)^2 \quad \lambda = -3$ - only eigenvalue

$\begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$, $\quad \begin{matrix} -2x + y = -3x \\ -x - 4y = -3y \end{matrix} \Big| \Rightarrow y = -x \quad$ Use $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

- only eigenvector

So $\quad e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is one soln.

To find another, find a generalized eigenvector: $A\vec{v} = -3\vec{v} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3x + 1 \\ -3y - 1 \end{pmatrix}$, $\quad \begin{matrix} -2x + y = -3x + 1 \\ -x - 4y = -3y - 1 \end{matrix} \Big| \Rightarrow y = -x + 1$

Use $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So $\quad e^{-3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$ is another soln.

Gen. soln: $\quad c e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + d e^{-3t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

(9) Consider the one-parameter family of differential equations $y' = y^2 - ay + 1$.

(a) Locate the bifurcation value (or values) of a.

Find equilibria: $y' = 0 = y^2 - ay + 1$, $y = \dfrac{a \pm \sqrt{a^2 - 4}}{2}$

2 roots if $-2 < a < 2$, 1 if $a = \pm 2$, 0 if $|a| > 2$.

So the bifurcation values are ± 2

(b) Draw the phase lines for values of a slightly smaller than, slightly larger than, and at the bifurcation value (or values), and describe the long-term behavior of each system. (If you couldn't do part (a), draw the phase lines and describe the behavior of $y' = -y^2 + 1$, $y' = -y^2$, and $y' = -y^2 - 1$.)

$a > 2$ — r_1, 1, r_2

$y > r_1 \Rightarrow y \to \infty$

$r_1 > y > r_2 \Rightarrow y \searrow r_2$

$r_2 > y \Rightarrow y \nearrow r_2$

$a = 2$ — 1

$y > 1 \Rightarrow y \nearrow \infty$

$y < 1 \Rightarrow y \nearrow 1$

~~$a > 2$~~ $-2 < a < 2$

$y \nearrow \infty$ for all y

$a = -2$ — -1

$y > -1 \Rightarrow y \nearrow \infty$

$y < -1 \Rightarrow y \nearrow -1$

$a < -2$ — r_1, -1, r_2

$y > r_1 \Rightarrow y \nearrow \infty$

$r_1 > y > r_2 \Rightarrow y \searrow r_2$

$r_2 > y \Rightarrow y \nearrow r_2$

OR: $y' = -y^2 + 1$ — 1, -1

$y > 1 \Rightarrow y \searrow 1$

$1 > y > -1 \Rightarrow y \nearrow 1$

$-1 > y \Rightarrow y \searrow -\infty$

$y' = -y^2$ — 0

$y > 0 \Rightarrow y \searrow 0$

$y < 0 \Rightarrow y \searrow -\infty$

$y' = -y^2 - 1$

$y \searrow -\infty$ for all y

EXTRA CREDIT (10 pts) The butterfly effect has been described as the idea that if a butterfly flaps its wings in Brazil on Tuesday, it can set off a tornado in Texas on Friday. Briefly explain this idea in mathematical terms.

Sensitive dependence on initial conditions: In a nonlinear system, a small ~~change~~ difference in solutions can lead to completely different behavior after a short time.