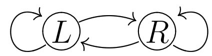
Math 6D Homework #4

Due in class Friday, Dec. 5.

1. The dynamical system $F(x) = -2x$ clearly has 0 as a fixed point and no other periodic points. But here is a "proof" that it has a period-two point. What is the flaw in the argument?

PROOF: Define the intervals $L = [-1, 0]$ and $R = [0, 1]$. The transition diagram is then:



Since LRL is a path, our theorem tells us that there's a fixed point x_0 for F^2 in the interval L whose itinerary is $LRLRLR\ldots$. Therefore x_0 has period two.

2. Let $G : [0, 1] \to [0, 1]$ be our old friend the doubling map,

$$G(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2, \\ 2x - 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

and $T : [0, 1] \to [0, 1]$ our older friend the tent map,

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2 - 2x & \text{if } 1/2 \le x \le 1. \end{cases}$$

(a) Establish sensitive dependence for G and T. Show that each point has neighbors arbitrarily nearby that eventually map at least 1/2 units apart.

(b) In fact, the doubling map G has a stronger property: Any two points x and y eventually map at least 1/2 unit apart (i.e., given $x \neq y$, there's a nonnegative integer n such that $dist(f^n(x), f^n(y)) \ge 1/2$). A map with this property is called *expansive*. Show that the tent map T is *not* expansive.

(c) Are G and T chaotic?

3. Let $F : [0, 1] \to [0, 1]$ be the logistic map defined by $F(x) = 4x(1 - x)$. Use an appropriate transition diagram to help show that F has a period-three point. What does Sharkovskii's theorem tell us about the periodic points of F?

4. (a) Can a continuous function on $\mathbb{R}$ have a periodic point of period 48 but not one of period 56? Explain.

(b) Can a continuous function on $\mathbb{R}$ have a periodic point of period 176 but not one of period 96? Explain.

5. Show that the union of two countable sets is countable.

6. Show that the middle-third Cantor set is uncountable. (HINT: ternary expansion.)

7. Consider the *middle-half Cantor set* $K(4)$ formed by deleting the middle half of each subinterval instead of the middle third.

(a) What is the box-counting dimension of $K(4)$?

(b) Consider the *parametrized tent map* T_a defined by

$$T_a(x) = \begin{cases} ax & \text{if } x \le 1/2 \\ a - ax & \text{if } 1/2 \le x. \end{cases}$$

Let S_a be the set of points in $[0, 1]$ that T_a never maps outside of $[0, 1]$. For what value of a is $S_a = K(4)$? (Recall that S_3 equals the usual middle-third Cantor set.)