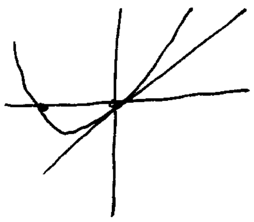
Math 6D Practice Midterm Solns

(1) Fixed pts. are solns. of $F(x)=x$, ie, $x=x^2+x$, or $0=x^2$, or $0=x$. So 0 is the only fixed pt. $F'(x)=2x+1$, so $F'(0)=1$, so we can't tell from the derivative whether 0 is attr. or rep. So let's think about it & look at the picture: the graph of F is a parabola with roots 0 & -1



If $x>0$, then $F^n(x)\to\infty$.

If $x<-1$, then $F(x)>0$, so $F^n(x)\to+\infty$

If $x=-1$, then $F(x)=0$, so it's eventually fixed

If $-1<x<0$, then $F(x)>x$ but still negative, so $F^n(x)\to 0$.

Thus 0 is attracting on the left & repelling on the right.

(2) a) $x=\frac{1}{x^2}$, or $x^3=1$ (since x can't be 0), so $x=1$ is the only fixed pt. $F'(x)=\frac{-2}{x^3}$, so $F'(1)=-2$, so 1 is a source.

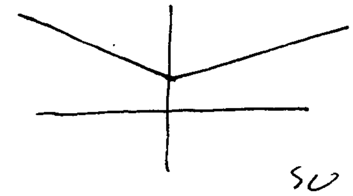
b) $x=1-x^2$, or $x^2+x-1=0$, or $x=\frac{-1\pm\sqrt{5}}{2}$ ← fixed pts. $F'(x)=-2x$, so $F'\left(\frac{-1\pm\sqrt5}{2}\right)=+1\pm\sqrt5\approx+1\pm2.23\approx 3.23$ or -1.23, so both fixed pts. are repelling

c) $x=x^2-1$, or $x^2-x-1=0$, or $x=\frac{1\pm\sqrt5}{2}$ ← fixed pts. $F'(x)=2x$, so $F'\left(\frac{1\pm\sqrt5}{2}\right)=1\pm\sqrt5\approx 3.23$ or -1.23, so both are repelling

(3) a) Fixed: Solve $x=-x^3$, so $x=0$ or $1=-x^2$ ← no soln. Only fixed pt. is 0. $F'(x)=-3x^2$, so $F'(0)=0$, so 0 is a sink.

$Per_2(F)=Fix(F^2)-Fix(F)$. $Fix(F^2)$ - solve $x=F^2(x)=F(-x^3)=x^9$, so $x=x^9$, so $x=0$ or $1=x^8$, so $x=\pm1$. 0 is fixed, so 1 & -1 are the period-two pts. $(F^2)'(1)=(F^2)'(-1)=F'(1)F'(-1)=-3\cdot3=-9$, so $\{1,-1\}$ is a repelling 2-cycle.

$Per_3(F)=Fix(F^3)-Fix(F)$ $Fix(F^3)$ - solve $x=F^3(x)=-x^{27}$, so $x=-x^{27}$, so $x=0$ or $1=-x^{26}$ ← no soln. 0 is a fixed pt., so ~~[illegible]~~ there are no period-3 pts.

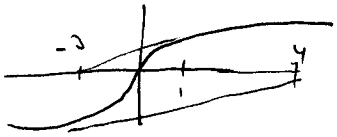
(3) b)  Fix F: solve ~~x~~ $x = F(x) = \left|\frac{1}{2}x\right| + \frac{1}{2}$

$F(x)$ is always > 0, so any fixed x must be > 0, so $\left|\frac{1}{2}x\right| = \frac{1}{2}x$. So solve $x = \frac{1}{2}x + \frac{1}{2}$, $\frac{1}{2}x = \frac{1}{2}$, or $x = 1$ is the only fixed pt. The slope at 1 is $\frac{1}{2}$, so it's attracting.

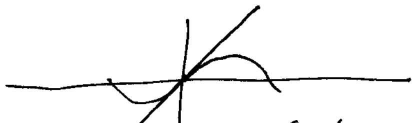
Again, since $F(x) > 0$ always, any periodic x must be > 0. So, if x is period-two, $x = F^2(x) = F\left(\frac{1}{2}x + \frac{1}{2}\right) = \frac{1}{4}x + \frac{1}{4} + \frac{1}{2}$, so $x = \frac{1}{4}x + 3/4$, or $3/4\,x = 3/4$, or $x = 1$, which was fixed, so there are no period-two pts. Similarly, there are no period-3 pts. (Another way to see that there are no periodic pts. is to realize that every pt. is attracted to the fixed pt. $x = 1$.)

(4) The only root of $\sqrt[3]{x}$ is $x = 0$. The Newton function is
$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\sqrt[3]{x}}{\frac{1}{3}x^{-2/3}} = x - 3x = -2x \quad (x \neq 0)$.

So $x_0 = 1$, $x_1 = -2$, $x_2 = 4$, $x_3 = -8$, $\ldots \to$ we get farther & farther from 0. The problem is that $f(x)$ is not differentiable at 0.

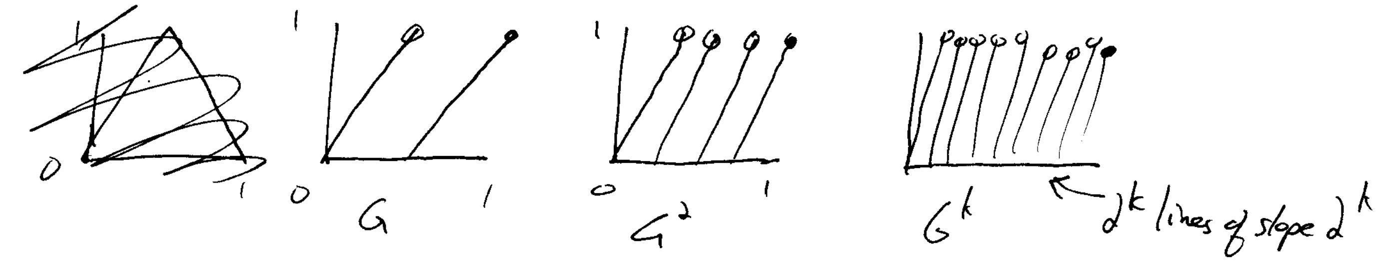


(5) No - points of period two must come in pairs.

(6) a) Look at the graph:  Since the slope is < 1 in absolute value near 0, 0 is a sink.

b) $G_c'(x) = c\cos x$, so $G_c'(0) = c$. 0 will be attracting if $|c| < 1$ & repelling if $|c| > 1$. If $c = 1$, we saw in part a) that 0 is attracting, & the same argument shows that 0 is attracting if $c = -1$.

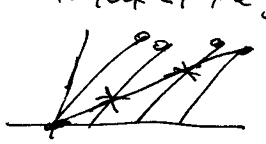
(7) a)



b) Fix(F): $x = 2x$ $(x < \frac{1}{2})$, so $x = 0$ ✓
or $x = 2x - 1$ $(\frac{1}{2} \le x \le 1)$, so $x = 1$ ✓

Both are repelling.

Period-2 pts – easier to look at the graph. We get 0 & 1, of course, and also two in the middle



, the eqn. for the second line is $F^2(x) = 4(x - \frac{1}{4})$ $(\frac{1}{4} \le x < \frac{1}{2})$,
so solve $x = 4x - 1$, or $x = 1/3$.

The other period-two pt. must be $F(1/3) = 2/3$. The slope of F^2 is 4,
so $(1/3, 2/3)$ is a repelling 2-cycle.

c) $1/10, 1/5, 2/5, 4/5, 3/5, 1/5, \ldots$

d) You don't have to be able to do this for your exam.
Write $x = \frac{p}{q}$ in lowest terms. F is just multiplication by 2 as long as the denominator is a multiple of 2, so keep applying F until you get $F^n(x) = \frac{p'}{q'}$, where q' is not a multiple of 2. Why must $\frac{p'}{q'}$ be periodic? HINT: there are only finitely many integers between 0 & q'.