1. (BABYLONIAN SQUARE ROOT METHOD) Here's a way, discovered by the ancient Babylonians, to approximate the square root of a positive number a. Start with an initial guess x_0. It could be that our guess is too big, i.e., $x_0 > \sqrt{a}$, and then we'll have $\frac{a}{x_0} < \sqrt{a}$, i.e., $\frac{a}{x_0}$ will be too small. On the other hand, if our guess x_0 is too small, then $\frac{a}{x_0}$ will be too big. Either way, if we average x_0 and $\frac{a}{x_0}$ (one of which is too big and one of which is too small), we should get a better guess, x_1. We can repeat this process over and over, getting (we hope) better and better estimates.

In other words, our Babylonian function $B_a(x)$ for finding the square root of a is

$$B_a(x) = \frac{x + \frac{a}{x}}{2}.$$

(If x is our current guess, then $B_a(x)$ is our next guess.)

Show that $\sqrt{a}$ is an attracting fixed point for B_a. What does this say about how well the method will work?

First, show it's fixed: $B_a(\sqrt{a}) = \dfrac{\sqrt{a} + \frac{a}{\sqrt{a}}}{2} = \dfrac{\sqrt{a}+\sqrt{a}}{2} = \dfrac{2\sqrt{a}}{2} = \sqrt{a}$ ✓

To show it's attracting, look at $B_a'(\sqrt{a})$.

$$B_a'(x) = \frac{1 - \frac{a}{x^2}}{2}, \text{ so } B_a'(\sqrt{a}) = \frac{1 - \frac{a}{(\sqrt{a})^2}}{2} = \frac{1-\frac{a}{a}}{2} = \frac{1-1}{2} = 0$$

$0 < 1$, so $\sqrt{a}$ is attracting.

This means that if your first guess is decent (close enough to $\sqrt{a}$), then the method will work, and successive guesses will get better & better.

2. Consider the *trough map* $T : [0,1] \to [0,1]$ defined by

$$T(x) = \begin{cases} 1 - 2x & \text{if } 0 \le x \le 1/2, \\ 2x - 1 & \text{if } 1/2 \le x \le 1. \end{cases}$$

(a) Find the first five terms on the orbit of $\dfrac{1}{11}$.

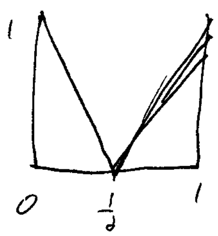
(b) Draw the graphs of T and T^2.

(c) Find and classify the fixed points and period-two points of T.

a) $F(\frac{1}{11}) = 1 - \frac{2}{11} = \frac{9}{11}$; $F(\frac{9}{11}) = \frac{18}{11} - 1 = \frac{7}{11}$; $F(\frac{7}{11}) = \frac{14}{11} - 1 = \frac{3}{11}$;

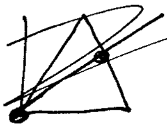
$F(\frac{3}{11}) = 1 - \frac{6}{11} = \frac{5}{11}$. So the orbit starts off $\frac{1}{11}, \frac{9}{11}, \frac{7}{11}, \frac{3}{11}, \frac{5}{11}, \ldots$

b)

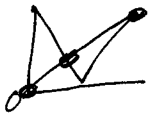


T

c) Fixed pts - use the graph



$\cancel{0 \text{ \& } x = 2x-1}$

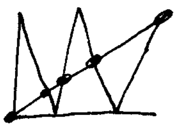


$\cancel{x = 0 \text{ \& } x = 1}$ $X = 1$ & $x = 1 - 2x$

$3x = 1$

$x = 1/3$

Period-two - use the graph



$1/3$ & 1 are fixed,

so solve $X = -4(x - \frac{1}{4}) = -4x + 1$

$5x = 1$

$x = \frac{1}{5}$

& $x = -4(x - \frac{3}{4}) = -4x + 3$

$5x = 3$

$x = 3/5$

So the 2-cycle is $\left(\frac{1}{5}, \frac{3}{5}\right)$.

3. (a) Let X be the space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Then the map $D : X \to X$ that sends each function to its derivative is a dynamical system on X. (That is, if $f(x)$ is an element of X, then $D(f(x)) = f'(x)$.)

What are the fixed points of D?

$\mathrm{Fix}(D)$ = all fns f with $f(x) = f'(x)$.

Solns. are $\{ce^t\}$, where c is any number.

(b) Now let Y be the subspace of X consisting of all polynomial functions, and let $D_Y : Y \to Y$ be the map that sends each polynomial to its derivative. (We call D_Y the *restriction of D to Y*.)

What are the possible long-term behaviors of points in Y? (Remember, a point in Y is a polynomial.)

Well, $D(ax+b) = a$, so $D^2(ax+b) = D(a) = 0$.

& $D^3(ax^2+bx+c) = 0$, etc.

In general, D^{n+1}(nth degree polynomial) $= 0$.

So every pt. in Y ~~g~~ is eventually fixed at 0.

4. Define *attracting fixed point.*

x_0 is an attracting fixed pt. for F if there is a nbhd. N_ε of x_0 such that $F^n(x) \to x_0$ for every x in N_ε.

(EXTRA CREDIT) Let $F : X \to X$ be a discrete dynamical system on some space X. Is it possible for F to have exactly four fixed points and F^3 to have exactly nine fixed points? If so, give an example of such an F and X. If not, explain why.

Not possible. $Per_3(F) = Fix(F^3) - Fix(F)$, so there would be $9-4=5$ period-3 pts. But period-3 points come in sets (3-cycles) of 3, so ~~they~~ their number must be a multiple of 3.