Math 6D Practice Midterm

1. Let $F(x) = x^2 + x$. Find and classify the fixed points. What are the possible long-term behaviors of points under F?

2. Find and classify the fixed points of the following functions.

(a) $F(x) = 1/x^2$.

(b) $F(x) = 1 - x^2$.

(c) $F(x) = x^2 - 1$.

3. (a) Let $F(x) = -x^3$. Find and classify the fixed points, the period-two points, and the period-three points.

(b) Same question, with $F(x) = |\frac{1}{2}x| + \frac{1}{2}$.

4. Let $f(x) = \sqrt[3]{x}$. What are the roots? What happens if you use Newton's method with initial guess $x_0 = 1$?

5. Is it possible for a discrete dynamical system F to have exactly three points of (least) period two?

6. (a) Let $G(x) = \sin x$. Clearly 0 is a fixed point for G. Is it attracting, repelling, or neither?

(b) For what values of the parameter c will the point 0 be an attracting fixed point for the function $G_c(x) = c \sin x$?

7. Let $G : [0, 1] \to [0, 1]$ be the *doubling map* defined by

$$G(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2, \\ 2x - 1 & \text{if } 1/2 \le x \le 1. \end{cases}$$

(a) Draw the graphs of G, G^2, and G^k.

(b) Find and classify the fixed points and period-two points of G.

(c) What is the orbit of the point $1/10$?

(d) (HARDER) Show that x is eventually periodic or eventually fixed if and only if x is rational (i.e., $x = p/q$, where p and q are integers).

(e) ASIDE: G is clearly not continuous as a function from the interval $[0, 1]$ to itself. However, if we glue together the points 0 and 1, we get a circle, and G *is* continuous as a function from this circle to itself. This is the same thing as saying that $G(x) = 2x \mod 1$ (i.e., $G(x)$ is the fractional part of $2x$), if that means anything to you. Anyway, you don't need to know any of this for your exam.