

GR HW 10/13/03 solns

① Lots of ways to do this. For example, could use $f(x) = x^{1/3}$ at $a=1$ in (i), $\ln x$ at $a=1$ in (ii), and $\frac{1}{\sqrt{x}}$ at $a=1$ or 4 in (iii).

② a) underestimate - we're missing $\frac{t^3}{3!} + \frac{t^4}{4!} + \dots$, which will be positive since $t > 0$.

b) $|B_{n+3}(t)| \leq \frac{M|t|^{n+3}}{(n+3)!}$, where M is Lip. for $(e^t)^{'''}=e^t$ on $[0, t]$.

This is biggest when $t = \frac{1}{2}$, so need $M \geq e^{\frac{1}{2}}$. Since $e^{\frac{1}{2}} \approx \sqrt{e} < \sqrt{4} = 2$, could use $M = 2$. So $|E_3(t)| \leq \frac{3|t|^3}{3!} = \frac{|t|^3}{2}$. Since $0 \leq t \leq \frac{1}{2}$,

$$\text{The error is } \leq \frac{|t|^3}{2} = \frac{1}{16}.$$

③ a) Taylor series is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$. Since $|x| \leq 1$, this is an alt. decr. series, so the error is less than the next term.

$$\text{So the error using } 1 - \frac{x^2}{2!} + \dots + \frac{x^{2k}}{(2k)!} \text{ is } \leq \frac{x^{2k+2}}{(2k+2)!} \leq \frac{1}{(2k+2)!}$$

$$(\text{If } n=2k, \text{ then } |E_n| \leq \frac{1}{(n+2)!})$$

b) solve $\frac{1}{(n+2)!} \leq \frac{1}{10^4}$, $(n+2)! \geq 10^4$, set $n = 6$

- solve $\frac{1}{(n+2)!} \leq \frac{1}{10^6}$, $(n+2)! \geq 10^6$, set $n = 8$

④ $|E_n(x_0)| \leq \frac{M|x_0|^{n+1}}{(n+1)!}$, where M is Lip. for $(e^x)^{(n+1)}=e^x$ on $[0, x_0]$ (or $(x_0, 0)$ if $x_0 < 0$). Take any $M \geq \max(1, e^{x_0})$. Then

$$|E_n(x_0)| \leq \frac{M|x_0|^{n+1}}{(n+1)!}, \text{ which } \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } x_0.$$

⑤ $\sum \frac{1}{x^p}$ converges $\Leftrightarrow \int_1^\infty \frac{1}{x^p} dx$ converges. $\int_1^\infty \frac{1}{x^p} dx = \left[\frac{1}{1-p} x^{1-p} \right]_1^\infty$

$$= \lim_{b \rightarrow \infty} \frac{1}{1-p} b^{1-p} - \frac{1}{1-p} = \begin{cases} \infty & \text{if } p < 1 \\ \frac{1}{1-p} & \text{if } p > 1 \end{cases}$$

$$\textcircled{6} \quad a) \sum \frac{1}{n^{3/2}} \rightarrow \text{conv. p-series}$$

b) conv, by alt. series test c) diverges - n^{th} term goes to $\pm \infty$

\textcircled{7} Converges by alt. series test (or, $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}$ is
conv. p-series, so it converges absolutely).

~~$$|E_n| \leq |\text{next term}| = \left| \pm \frac{1}{n^4} \right| = \frac{1}{n^4}$$~~

Solve $\frac{1}{n^4} < \frac{1}{1000}$, $n^4 > 1000$, $n > \sqrt[4]{1000} \approx 5.6$

So use $n=6$.

$$\begin{aligned} \text{So } S_{4m} &= 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} \pm .001 \\ &= \cancel{1.9468} \pm .001 \end{aligned}$$