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## FINAL EXAM

This exam is 8 pages long; check that you have all the pages. Show your work. Correct answers with no justification may receive little or no credit. No calculators, notes, or books are allowed. No uncalled-for simplification is required. Use the backs of pages if you run out of space, and make sure that I can find your answers.

Please ask questions if you're confused!

| PROBLEM | POINTS | SCORE |
| :---: | :---: | :---: |
| 1 | 15 |  |
| 2 | 20 |  |
| 3 | 20 |  |
| 4 | 10 |  |
| 5 | 10 |  |
| 6 | 15 |  |
| 7 | 10 |  |
| EXTRA CREDIT | 5 |  |
| TOTAL | 100 |  |

(1) (15 pts)
(a) Does $\sum_{n=1}^{\infty} \frac{1}{n+\frac{1}{n}}$ converge or diverge? Explain why.
(b) Why does $\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n+\frac{1}{n}}$ converge? How close is the second partial sum to the actual sum of the series? (I want an actual number.)
(2) (20 pts) My butler Buttlesworth is retiring this year. To reward him for years of faithful service, I plan to give him $\$ 100,000$ each May 9, starting today, until he dies. I want to put all the necessary money in a bank account (earning $10 \%$ interest, compounded annually) right now.
(We are supposing that every May 8, I receive the interest for the previous year (exactly 0.10 times the amount in the bank). Thus, if I deposit $\$ 100$ today and don't spend any of it over the next year, then on May 8, 2004, I will have $\$ 100(1.10)$.)
(a) If I think that Buttlesworth is going to live another $19 \frac{1}{2}$ years (meaning that he gets 20 payments of $\$ 100,000$ - one today, one in one year, $\ldots$, and one in nineteen years), how much money do I need to deposit in the bank account right now to cover his payments? (If I thought he'd die in 6 months, I'd need to deposit $\$ 100,000$ right now to cover today's payment.)
(b) If I think that Buttlesworth is actually a robot who will live forever, how much money do I need to deposit in the bank account right now to cover his payments?
(3) (20 pts) Out of the kindness of Prof. Klotz' and Prof. Wiseman's hearts, here are the derivatives of $f(x)=\ln x$ :

$$
f^{\prime}(x)=\frac{1}{x}, f^{\prime \prime}(x)=\frac{-1}{x^{2}}, f^{\prime \prime \prime}(x)=\frac{2!}{x^{3}}, f^{(4)}(x)=\frac{-3!}{x^{4}}, \ldots, f^{(n)}(x)=\frac{(-1)^{(n-1)}(n-1)!}{x^{n}}, \ldots
$$

(a) What is the Taylor series with base point $a=2$ for $\ln x$ ?
(b) What is the radius of convergence?
(c) What is the interval of convergence?
(3) (continued)
(d) If you use the third-degree Taylor polynomial to approximate $\ln x$ for $1<x<3$, what's the most you could possibly be off by?
(4) (10 pts) Using only the definition of convergence of series, determine whether the series $\sum_{n=1}^{\infty}\left(\frac{n+1}{n}-\frac{n+2}{n+1}\right)$ converges or diverges. If it converges, find the sum.
(5) (10 pts) Find the first three nonzero terms of the power series solution of the form $\sum c_{n} x^{n}$ for the differential equation $y^{\prime}=x y$ with initial condition $y(0)=1$.
(6) (15 pts) Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false. (Each $a_{n}$ is a (not necessarily positive) real number.)
(a) If the terms, $a_{n}$, of a series tend to zero as $n$ increases, then the series $\sum a_{n}$ converges.
(b) If $\sum a_{n} x^{n}$ diverges when $x=1$, then it diverges when $x=-2$.
(c) If $\sum a_{n} x^{n}$ converges when $x=7$, then it converges when $x=-9$.
(d) If $\sum a_{n}$ converges, then $\sum(-1)^{n} a_{n}$ converges.
(e) Let $s_{1}, s_{2}, \ldots$ be a sequence with $\lim _{n \rightarrow \infty} s_{n}=L$ ( $L$ is a real number). If $s_{10}=s_{20}=s_{30}=\ldots=12$, then $L=12$.
(7) (10 pts) We want to study the populations of whales in the ocean and of mold on the bread in my refrigerator. We know that whales live in the vast, vast ocean and have plenty of room and lots of delicious plankton to eat. At the same time, though, they're so spread out that if there aren't very many of them, they won't be able to find each other in order to breed.

We also know that mold reproduces asexually (i.e., a mold cell doesn't need another mold cell to reproduce). On the other hand, once the mold covers the bread, there'll be no place else for it to live.

Differential equations modeling the two populations $P(t)$ and $Q(t)$ are given below. Which is which? That is, is $P(t)$ the whale population at time $t$ and $Q(t)$ the mold population, or is it the other way around? Explain your reasoning. ( $k_{1}$ and $k_{2}$ are positive constants.)

$$
\frac{d P}{d t}=k_{1} P(P-10,000) \quad \frac{d Q}{d t}=k_{2} Q(10,000-Q)
$$

EXTRA CREDIT (5 pts) Explain why it's important for us to be able to estimate how well the $n$th partial sum of a convergent Taylor series approximates the true sum of the series.

