1. For each of the following, show how you can estimate the magnitude of the error in approximating the given quantities using a third-degree Taylor polynomial.
(a) $0.5^{1 / 3}$
(b) $\ln (1.5)$
(c) $1 / \sqrt{3}$
2. Suppose that you approximate $f(t)=e^{t}$ by a Taylor polynomial of degree 2 about $t=0$ on the interval [0, 0.5].
(a) Reasoning informally, say whether the approximation is an overestimate or an underestimate.
(b) Estimate the magnitude of the largest possible error.
3. (a) Give a bound for the maximum possible error for the $n$ th-degree Taylor polynomial about $x=0$ approximating $\cos x$ on the interval $[0,1]$.
(b) What degree Taylor polynomial about $x=0$ do you need to calculate $\cos 1$ to with 0.0001 ? To within $10^{-6}$ ?
4. Show that the Taylor series about 0 for $e^{x}$ converges to $e^{x}$ for every $x$. Do this by showing that the error $E_{n}(x) \rightarrow 0$ as $n \rightarrow \infty$.
5. Use the integral test to show that the $p$-series $\sum 1 / n^{p}$ converges if $p>1$ and diverges if $p \leq 1$. (This completes a problem from an earlier assignment.)
6. Which of the following series converge? (a) $\sum \frac{1}{\sqrt{2 n^{3}}}$ (b) $\sum \frac{(-1)^{n-1}}{2 n+1}$ (c) $\sum \frac{(-1)^{n-1} 2^{n}}{n^{2}}$
7. Explain why the series $1-\frac{1}{2^{4}}+\frac{1}{3^{4}}-\frac{1}{4^{4}}+\cdots$ converges and find its sum correct to within 0.001 .
8. If $\sum a_{n}$ diverges, why does it follow that $\sum\left|a_{n}\right|$ diverges? (Thus there is no concept of absolute or conditional divergence, as there is for convergence.)
9. A conditionally convergent series, like $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ can be arranged to converge to any number you want, say $\pi$. Explain how this may be done, as follows.
(a) Why is it possible to add up enough positive terms $\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)$ to eventually exceed $\pi$ ?
(b) Supposing this to have been done (with no more terms than necessary), why is it possible to continue with negative terms $\left(-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots\right)$ until the sum in part (a) becomes less than $\pi$ ?
(c) Add more positive terms to get the sum above $\pi$ again, then more negative terms to bring it below $\pi$, and continue in this way indefinitely (always using just enough terms to do the job). Do we run out of positive terms? Of negative terms? Is every term of the original series eventually used? Why does this process produce sums that converge to $\pi$ ?
10. Explain why the series in problem 9 can be rearranged to diverge, as follows.
(a) Why is it possible to add up enough positive terms to exceed 10 ? Why is it possible to continue with enough negative terms to bring the sum below $-10^{2}$ ?
(b) Add more positive terms to get the sum above $10^{3}$, then more negative terms to bring it below $-10^{4}$, and continue in this way indefinitely. Do we run out of positive terms? Of negative terms? Is every term of the original series eventually used? Why does this process produce sums that have no limit?
11. What happens if you rearrange the terms of an absolutely convergent series?
