Math 6B(1) Homework #4

Due in class Friday, Oct. 3.

1. Which of the following are power series?

(a) $\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$ (b) $1 + x + (x-1)^2 + (x-1)^3 + (x-1)^4 + \cdots$ (c) $x^7 + x + 2$

2. Find the domain for each of the following functions:

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ (b) $f(x) = \sum_{n=1}^{\infty} \left(\frac{-x}{n}\right)^n$ (c) $f(x) = x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \cdots$

3. Suppose that the domain of $f(x) = c_0 + c_1(x+1) + c_2(x+1)^2 + \cdots$ includes $x = 1$. Does it necessarily include $x = 0$? What about $x = -3$?

4. Find the interval of convergence of the power series $1 - \frac{x^2}{3} + \frac{x^4}{9} - \frac{x^6}{27} + \cdots$. (HINT: You can't use the ratio test on this series (why not?). Instead, consider the series $1 - \frac{t}{3} + \frac{t}{9} - \frac{t}{27} + \cdots$. What does the interval of convergence of this series tell you about the interval of convergence of the original series?)

5. Let $f(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$. Find $\int_0^x f(t)\,dt$, where x is any point in the domain of f.

6. Let $g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$.

(a) Is this a power series?

(b) Show that the series converges absolutely for every x (so the domain of g is all of $\mathbb{R}$).

(c) It's reasonable to guess that g is differentiable everywhere and that

$$g'(x) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{\sin nx}{n^2}\right) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}.$$

Show that this is not the case, by showing that the proposed series for $g'(x)$ diverges at $x = 0$.

(d) Be extra thankful that we *can* differentiate a power series termwise within its interval of convergence, because infinite series aren't usually so well behaved, as you've just shown.

7. Find the radius of convergence of the Maclaurin series for $\ln(1-x)$.

8. By recognizing the following as a Taylor series evaluated at a particular value of x, find the sum of $1 + \frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \cdots + \frac{2^n}{n!} + \cdots$.

9. Solve the following for x: $1 + x + x^2 + x^3 + \cdots = 5$.

10. Suppose that all the derivatives of the function f exist at 0, and that the Taylor series for f about $x = 0$ is $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots$. Find $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(10)}(0)$.