Math 61 **MIDTERM SOLUTIONS** **October 6, 2004**

1. Explain what the Alabama paradox is, and why it can't occur if we use the Jefferson method of apportionment.

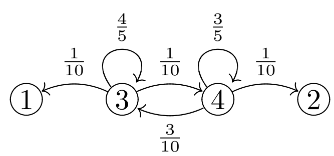
The Alabama paradox occurs when, under some apportionment method, at house size h a state gets n seats, but if the house size is increased to $h+1$, that same state gets only $n-1$ seats. This can't occur under the Jefferson method, because that method works by giving each state one seat, then handing out the remaining seats one at a time to the "most deserving" state. Thus increasing the house size means that there's one extra seat to hand out, so some lucky state gets one more, and no state loses one. (Another way to look at the Jefferson method is in Beltrami. Divide each state's population by a certain number, then round down to get their apportionment. Increasing the house size just means a smaller divisor, so no state can lose a seat.)

2. For reasons that I've never fully understood, when I was growing up my father would often say to me, "Fun's fun, but who wants to die laughing?" Since fun *is* fun, let's calculate the probability that I'll die laughing. Let's say that every minute, I'm either laughing or not. If I'm laughing now, there's a 4/5 probability that I'll be laughing the next minute, a 1/10 probability that I'll be alive but not laughing the next minute, and a 1/10 probability that I'll be dead the next minute (so I died laughing). If I'm alive but not laughing now, there's a 3/10 probability that I'll be laughing the next minute, a 3/5 probability that I'll be alive but not laughing the next minute, and a 1/10 probability that I'll be dead the next minute.

(a) What is the probability that I'll die laughing, given that I'm not laughing now?

(b) What is my expected lifetime, given that I'm not laughing now? What fraction of my remaining minutes can I expect to spend laughing?

Construct the following Markov transition graph (state 1 is "died laughing," state 2 is "died not laughing," state 3 is "alive and laughing," and state 4 is "alive and not laughing"):



The transition matrix is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/10 & 0 & 4/5 & 1/10 \\ 0 & 1/10 & 3/10 & 3/5 \end{bmatrix} = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$.

(a) We want the probability that I get absorbed into state 1, given that I started in state 4. This is the bottom left entry of the matrix B, defined by $B = R + QB$. A little algebra gives 3/5.

(b) The fundamental matrix T, defined by $T = (I-Q)^{-1}$, is $\begin{bmatrix} 8 & 2 \\ 6 & 4 \end{bmatrix}$. So I can expect to live 6+4=10 minutes, during 6 of which I'll be laughing.

2. The registrar is scheduling final exams, and he needs your help. To begin, he has constructed an undirected graph G, defined as follows:

- The vertices are the classes being offered this semester. (So Math 61 is one vertex, and Bio 1 is another.)
- There's an edge between two vertices (classes) if there's at least one student who is enrolled in both classes.

Let's say that there are ten possible exam times (morning, afternoon, and evening on days one, two, and three, and morning on day four). Assign each exam time a different color. Then feasible exam schedules (that is, schedules under which no student has two finals scheduled at the same time) correspond to colorings of the graph (that is, a choice of colors for the vertices such that no two adjacent vertices are the same color).

(a) The registrar would prefer a schedule under which no student has three exams in a 24-hour period. *Given only the graph G* (that is, *not* the list of enrollments that generated it), can you tell whether a given schedule/coloring satisfies this condition? If so, how? If not, are there any circumstances under which you could answer either yes or no for sure?

(b) Okay, now forget about part (a). The registrar also wants to let as many students as possible leave early, that is, to minimize the number of students taking a final on the last day (during exam time 10). Formulate, *but do not solve*, an integer programming problem whose solution will tell him when to schedule each class's exam.

Here are some useful facts and notation:

(i) There are ten exam times, $k = 1, \ldots, 10$.
(ii) There are N different classes, $i = 1, \ldots, N$.
(iii) Each class must be scheduled for exactly one exam time.
(iv) No two classes whose vertices in G are adjacent can be scheduled for the same exam time.
(v) There are e_i students enrolled in class i.
(vi) Classes $1, \ldots, M$ have enrollments of 40 or more. The other classes have enrollments under 40.
(vii) No two classes can have their exams in the same classroom at the same time.
(viii) There are C classrooms available.
(ix) Any classroom can hold any class of under 40 students, but only B of these classrooms can hold a class of 40 or more students.

HINTS: What function are you trying to optimize? What are your constraints? Some useful notation:

$$x_{ik} = \begin{cases} 1, \text{ if class } i \text{ is scheduled for exam time } k, \\ 0 \text{ otherwise.} \end{cases}$$

and

$$a_{ij} = \begin{cases} 1, \text{ if class } i \text{ and class } j \text{ are adjacent in } G, \\ 0 \text{ otherwise.} \end{cases}$$

(a) We can't tell, in general, although we may be able to tell that the condition *is* satisfied. If a student has three exams in 24 hours (at times/colors n, $n+1$, and $n+2$), then there will be edges between some vertex v_1 colored n and a vertex v_2 colored $n+1$, between v_2 and a vertex v_3 colored $n+2$, and between v_1 and v_3. So if this doesn't happen for any n, then the condition is satisfied. On the other hand, if it *does* happen, then we can't say. It could be that Larry is in classes v_1 and v_2, while Moe is in classes v_2 and v_3, which would give the same result in the graph, but wouldn't cause a scheduling problem.

(b) We want to choose the x_{ik}'s to minimize the sum $\sum_{i=1}^{N} x_{i10} e_i$, given the constraints

- **For all i, $\sum_{k=1}^{10} x_{ik} = 1$ (from (iii)).**
- **$\sum_{k=1}^{10} \sum_{i<j} a_{ij} x_{ik} x_{jk} = 0$ (from (iv)).**
- **For all k, $\sum_{i=1}^{N} x_{ik} \leq C$ (from (vii)).**
- **For all k, $\sum_{i=1}^{M} x_{ik} \leq B$ (from (vii) and (ix)).**