1. Consider the ODE

$$(t^2+3)y'' - 7ty' + 16y = 0.$$

(a) Find the first three nonzero terms of each of two linearly independent solutions of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

$$y = a_0 + a_1 t + a_2 t^2 + \cdots$$
$$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots$$
$$y'' = 2a_2 + 3\cdot 2a_3 t + 4\cdot 3 a_4 t^2 + 5\cdot 4 a_5 t^3 + \cdots$$

So we have

$$2a_2 t^2 + 3\cdot 2 a_3 t^3 + 4\cdot 3 a_4 t^4 + 5\cdot 4 a_5 t^5 + \cdots$$
$$+ \ 3\cdot 2a_2 + 3\cdot 3\cdot 2 a_3 t + 3\cdot 4\cdot 3 a_4 t^2 + 3\cdot 5\cdot 4 a_5 t^3 + \cdots$$
$$+ \ -7a_1 t - 14 a_2 t^2 - 7\cdot 3 a_3 t^3 - 7\cdot 4 a_4 t^4 - \cdots$$
$$+ \ 16 a_0 + 16 a_1 t + 16 a_2 t^2 + \cdots$$
$$= 0$$

constant term: $6a_2 + 16a_0 = 0, \quad a_2 = \frac{-8}{3} a_0$

t term: $18a_3 - 7a_1 + 16a_1 = 0, \quad a_3 = -\frac{1}{2} a_1$

t^2 term: $2a_2 + 36a_4 - 14a_2 + 16a_2 = 0, \quad 36a_4 = -4a_2 = \frac{32}{3} a_0, \quad a_4 = \frac{8}{27} a_0$

t^3 term: $6a_3 + 60a_5 - 21a_3 + 16a_3 = 0, \quad 60a_5 = -a_3 = \frac{1}{2} a_1, \quad a_5 = \frac{1}{120} a_1$

$$\text{So } y = a_0 + a_1 t + \frac{-8}{3} a_0 t^2 + \frac{-1}{2} a_1 t^3 + \frac{8}{27} a_0 t^4 + \frac{1}{120} a_1 t^5 + \cdots$$

$$= a_0 \underbrace{\left(1 - \frac{8}{3} t^2 + \frac{8}{27} t^4 + \cdots\right)}_{\text{soln } y_1} + a_1 \underbrace{\left(t - \frac{1}{2} t^3 + \frac{1}{120} t^5 + \cdots\right)}_{\text{soln } y_2}$$

(b) What can you say about the intervals of convergence of the solutions?

$y'' - \frac{7t}{(t^2+3)} y' + \frac{16}{(t^2+3)} y = 0$. Interval of conv. of solns. at least as big as int. of conv. of coefficients, which blow up at $\pm\sqrt{3}\, i$. So int. of conv. is at least $(-\sqrt{3}, \sqrt{3})$.

2. Show, from the definition of the Laplace transform, that $\mathcal{L}[y'(t)] = s\mathcal{L}[y(t)] - y(0)$.

$\mathcal{L}[y'] = \int_0^\infty y' e^{-st}\,dt$ Integrate by parts: $\int u\,dv = uv - \int v\,du$

Here $dv = y'\,dt$ $u = e^{-st}$

$v = y$ $du = -se^{-st}\,dt$

$$= e^{-st} y \Big|_0^\infty - \int_0^\infty -y\, s e^{-st}\,dt$$

$$= \underbrace{\lim_{a\to\infty} e^{-sa} y(a)}_{0 \text{ for } s>0, \text{ since } y \text{ grows at most exp}} - e^{-s0} y(0) + s\int_0^\infty y e^{-st}\,dt$$

$$= -y(0) + s\,\mathcal{L}[y]$$

3. Find the general solution of the system $x' = x + 2y$, $y' = 2x + y$.

$$\begin{aligned} x' &= x + 2y \\ y' &= 2x + y \end{aligned}, \quad \text{or} \quad \vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x}$$

Find the eigenvalues & eigenvectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$:

char. poly. is $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda + 1 - 4$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

So eigenvalues are $3, -1$

$\underline{\lambda = 3}$ $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x} = \vec{0}$, $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$, so $-2x + 2y = 0$, or $y = x$

An eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\underline{\lambda = -1}$ $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x} = \vec{0}$, $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, so $2x + 2y = 0$, or $y = -x$.

An eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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So the gen. soln. is $c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

In the long term, almost every soln. goes off to ∞, approaching the line $y = x$ asymptotically. The only exceptions are solns starting on the line $y = -x$, which go to $\vec{0}$.

4. Suppose that the human population $y(t)$ in a large forest is growing too rapidly, and the sasquatches decide to give out hunting licenses that allow a total of h humans to be eaten per day over a 30-day period. A model for such a situation is

$$y'(t) = ky(t) - \begin{cases} h, & 0 \leq t \leq 30 \\ 0, & t \geq 30, \end{cases}$$

where k is a positive constant describing the natural growth rate of the human population.

Find $\mathcal{L}[y(t)]$, assuming that $y(0) = A$. (You don't need to solve for $y(t)$.)

$$y' = ky - h + u_{30}(t) \cdot h$$

Take the Laplace transform of each side:

$$sY(s) - \underbrace{y(0)}_{=A} = kY(s) - \frac{h}{s} + \frac{he^{-30s}}{s}$$

$$(s-k)Y(s) = A + \frac{h}{s}\left(e^{-30s} - 1\right)$$

$$Y(s) = \frac{1}{s-k} \cdot \left(A + \frac{h}{s}\left(e^{-30s} - 1\right)\right)$$