1. Solve the IVP

$$(2y+t)y' + y = 0, \quad y(-5) = \frac{9}{2}.$$

On what t-interval does this solution exist?

Since $\frac{d}{dt}(2y+t) = 1 = \frac{d}{dy}(y)$, the eqn. is exact ~~separable~~

$$(2y+t)y' + y = 0$$

$$(y^2+ty)' = 0$$

$$y^2+ty = C$$

Plug in $y(-5) = \frac{9}{2}$: $\frac{81}{4} - \frac{45}{2} = C$, $-\frac{9}{4} = C$

$$y^2 + ty + \frac{9}{4} = 0$$

$$y = \frac{-t \pm \sqrt{t^2-9}}{2}$$

Since $y(-5) = \frac{9}{2}$, $y = \frac{-t + \sqrt{t^2-9}}{2}$

The soln exists for $-\infty < t < -3$.

2. The tank holding your dorm's drinking water has a capacity of 100 gallons and is half full of fresh water. A pipe is opened that lets raw sewage enter the tank at the rate of 4 gallons per minute. At the same time, a drain is opened to allow the mixture to leave the tank at the rate of 2 gallons per minute. If the sewage contains 10 grams of potassium per gallon, what is the concentration of potassium in the tank the instant before it overflows?

The amt. of ~~water~~ fluid in the tank at time t is $50+2t$ $(0 \le t \le 25)$.
If $P(t)$ is the amt. of potassium in the tank at time t, then we have

$$P' = 40 - \frac{2P}{50+2t} = 40 - \frac{P}{25+t}, \text{ or}$$

$$P' + \frac{P}{25+t} = 40.$$ This is linear, so we look for an integrating factor $\mu(t)$.

Want $\mu P' + \frac{\mu P}{25+t} = (\mu P)'$, or $\mu' = \frac{\mu}{25+t}$, or

$$\frac{d\mu}{\mu} = \frac{1}{25+t}, \text{ or } \ln|\mu| = \ln|25+t| + C$$

$$\mu = \pm e^C(25+t)$$

Simplest: $\mu = 25+t$

So $(25+t)P' + P = (25+t)40$, or

$$((25+t)P)' = 1000+40t, \text{ so}$$

$$(25+t)P = 1000t + 20t^2 + C.$$

Since $P(0)=0$, $25 \cdot 0 = C$, or $C=0$.

So ~~$P(t)$~~ $(25+t)P = 1000t + 20t^2$, or

$$P = \frac{1000t + 20t^2}{25+t}$$

The tank overflows for $t > 25$, so we want

$$P(25) = \frac{1000 \cdot 25 + 20 \cdot 25^2}{2 \cdot 25} = \frac{1000 + 20 \cdot 25}{2} = 500 + 10 \cdot 25$$

$$= 750$$

3. Many chemical reactions can be viewed as interactions between two molecules that undergo a change and result in a new product. The rate of a reaction, therefore, depends on the number of interactions or collisions, which in turn depends on the concentrations of both types of molecules. Consider the simple reaction $A + B = Y$, in which one molecule of substance A combines with one molecule of substance B to create one molecule of substance Y.

Let's designate the initial number of molecules of A and B by constants α and β, respectively. Let $y(t)$ denote the number of molecules of Y at time t. Which of the following is the most reasonable model for $y(t)$? Explain. (Here, k is a positive constant.)

(a) $y' = k\alpha\beta$
(b) $y' = ky$
(c) $y' = k(\alpha - y)(\beta - y)$
(d) $y' = k[(\alpha - y) + (\beta - y)]$
(e) $y' = k\alpha\beta y$
(f) $y' = k\alpha\beta y(1 - y)$

c). y' is jointly proportional to ~~to~~ the amt. of A & the amt. of B, since the same is true of the number of collisions between atoms of A and atoms of B. Since each molecule of Y takes one of A and one of B, the amt. of A at time t is $\alpha - y(t)$, and similarly for B.

4. If a solution of the ODE
$$e^t y'' - t^2 y' + (\cos t^2) y = 0$$
is tangent to the t-axis at any point, then it must be identically zero. Why?

We can rewrite the ODE as $y'' - \frac{t^2}{e^t} y' + \frac{\cos t^2}{e^t} y = 0$.

The coefficients are continuous for all t, so solutions exist and are unique for all t.

If the soln. y is tangent to the t-axis at $t = t_0$, then $y(t_0) = y'(t_0) = 0$, and the unique soln. satisfying these initial conditions is $y \equiv 0$.

5. Find the general solution of the ODE

$$y'' + 8y' + 25y = e^t.$$

What's the long-term behavior?

First, solve the associated homogeneous ODE $y''+8y'+25y=0$. The char. poly is $r^2+8r+25$, with roots $r = \frac{-8 \pm \sqrt{64-100}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$.

So the gen. soln. is $ce^{-4t}\cos 3t + de^{-4t}\sin 4t$.

Next, find a particular soln. y_p. Guess $y_p = Ae^t$ & plug in:

$$Ae^t + 8Ae^t + 25Ae^t = e^t$$
$$34Ae^t = e^t$$
So $A = 1/34$, & $y_p = 1/34\, e^t$.

So the gen. soln. is $ce^{-4t}\cos 3t + de^{-4t}\sin 3t + 1/34\, e^t$.