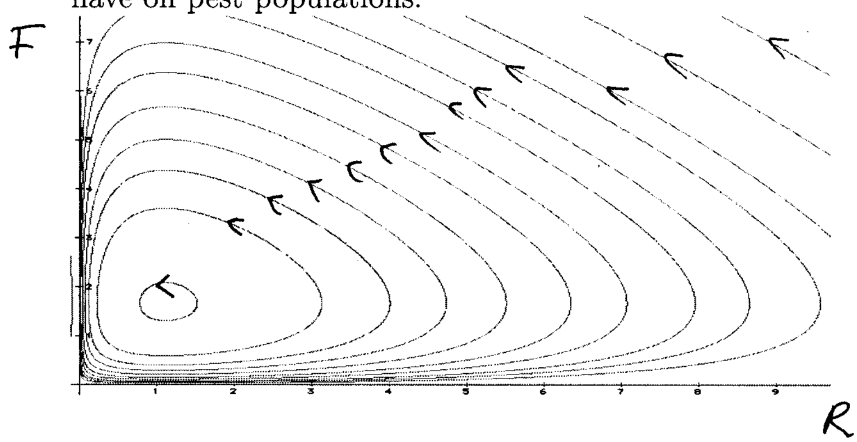
(1) (20 pts) Pesticides that kill an insect species not only are bad for the environment, but they can also be inefficient at controlling pest species. Suppose that a pest insect species in a particular field has population $R(t)$ at time t, and suppose that its primary predator is another insect species with population $F(t)$ at time t. Suppose that the populations of these species are accurately modeled by the system

$$\begin{aligned} R' &= 2R - 1.2RF \\ F' &= -F + 0.9RF \end{aligned}$$

Finally, suppose that at time $t = 0$ a pesticide is applied to the field, reducing both the pest and predator populations to very small but nonzero numbers.

(a) Using the figure below, which shows the phase portrait for this system, predict what will happen to the population of the pest species as t increases.

(b) Senator Haddock needs your help again. Write a paragraph, in nontechnical language, explaining to him the possibility of the paradoxical effect that pesticide application can have on pest populations.



a) The pest population will oscillate: It starts small, grows ENORMOUS, then gets small, then grows, - - -

b) Dear Sen. Haddock,

Be careful with the pesticide, as you are dealing with forces beyond your imagination. The only thing keeping the pest population under control is the predators. If you kill off the predators, the pests will come back in greater numbers than ever - the plague of locusts will be revisited upon you a thousand-fold. If what you're worried about is the <u>peak</u> number of pests, then you shouldn't use pesticide. If you just want to reduce the pest numbers temporarily (say until the next election), then go ahead and spray pesticide. But if you sow the wind, you shall reap the whirlwind.

Sincerely,

Jim

2) (15 pts) Find the general solution of $\mathbf{x}' = \begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix} \mathbf{x}$.

The only eigenvalue is $\lambda = -4$ & the only eigenvector is $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

So we need to find a generalized eigenvector $\vec{x}_0 = \begin{pmatrix} x \\ y \end{pmatrix}$, ie,

$$\begin{pmatrix} -8 & -1 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -4\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

$$\begin{aligned} -8x - y &= -4x + 1 \\ 16x &= -4y - 4 \end{aligned}$$

The simplest solution is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

Thus the general solution is

$$c_1 e^{-4t}\begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{-4t}\left(t\begin{pmatrix} 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right)$$

(3) (10 pts) Consider the system of ODEs $\mathbf{x}' = \mathbf{P}\mathbf{x}$. Let $\mathbf{x}_1(t)$ be the solution satisfying the initial condition $\mathbf{x}_1(7) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2(t)$ the solution satisfying $\mathbf{x}_2(7) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Do you have enough information to find a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(7) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$? If so, what is it? If not, why not?

Yes - the soln. is $\vec{x}(t) = 2\vec{x}_1(t) + \vec{x}_2(t)$.

Since $\vec{x}' = P\vec{x}$ is linear homogeneous, any linear combination of solns. is a soln, and $\vec{x}(7) = 2\vec{x}_1(7) + \vec{x}_2(7) = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

(Section 2 Only)

(4) (10 pts) Consider the initial value problem

$$y'' + ay' + by = \sin t, \; y'(0) = 0, \; y(0) = 0.$$

You are given that $f(t)$ is the solution of the initial value problem

$$y'' + ay' + by = \delta_0(t), \; y'(0) = 0, \; y(0) = 0,$$

where δ_0 is the Dirac delta "function." What is the solution of the original problem?

$$y(t) = f(t) * \sin(t) = \int_0^t f(t - t_0)\sin(t_0)\, dt_0$$

Because: $\mathcal{L}(y'' + ay' + by) = (s^2 + as + b)Y(s)$

& $\mathcal{L}(\delta_0(t)) = 1$, so $f = \mathcal{L}^{-1}\left(\frac{1}{s^2+as+b}\right)$ is the soln. to $y'' + ay' + by = \delta_0(t)$.

The same way, $\mathcal{L}^{-1}\left(\frac{1}{s^2+as+b} \cdot \mathcal{L}(\sin t)\right)$ is the soln. to $y'' + ay' + by = \sin t$, and $\mathcal{L}^{-1}\left(\frac{1}{s^2+as+b} \cdot \mathcal{L}(\sin t)\right)$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2+as+b}\right) * \mathcal{L}^{-1}(\mathcal{L}(\sin t))$$

$$= f(t) * \sin(t)$$

(5) (10 pts) Consider the two systems of differential equations

(a) $\begin{aligned} x' &= 0.3x - 0.1xy \\ y' &= -0.1y + 2xy \end{aligned}$ (b) $\begin{aligned} x' &= 0.3x - 3xy \\ y' &= -2y + 0.1xy \end{aligned}$

One of these systems refers to a predator-prey system with very lethargic predators – predators who seldom catch prey but who can live for a long time on a single prey (for example, boa constrictors). The other system refers to a very active predator that requires many prey to stay healthy (such as a small cat). The prey in each case is the same. Identify which system is which and justify your answer.

~~If $x' = ax - by$, $y' =$~~ If $\begin{aligned} x' &= ax - bxy \\ y' &= -cy + dxy, \end{aligned}$ then b is a measure of how likely the prey is to be eaten if it encounters a predator – big b means very likely, so an active predator. Similarly, c is a measure of how quickly a predator dies without food – big c means it dies quickly. The first system has small b & small c, so it probably corresponds to the lethargic predator. The second system has large b and large c, so it corresponds to the active predator.

(6) (10 pts) Consider the initial value problem

$$e^t y''' + (\ln t)y' + (\sin t)y = 0, \; y'(1) = 0, \; y(1) = 7.$$

For what values of t can you be sure that the solution exists? Why?

This corresponds to $y''' + \frac{\ln t}{e^t} y' + \frac{\sin t}{e^t} y = 0, \; y'(1) = 0, \; y(1) = 7.$

$\frac{\sin t}{e^t}$ is continuous & differentiable everywhere, but $\frac{\ln t}{e^t}$ is cts. & diff. only ~~on intervals~~ for $t > 0$ (since $\ln t$ doesn't even exist for $t \leq 0$). So our soln. exists for

$0 < t < \infty$.

(7) (10 pts)

(a) Rewrite $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$ as a series whose generic term involves x^n rather than x^{n-2}.

(b) Assume that the solution to $y'' - xy = 0$ is

$$a_0 + a_1 x + \frac{a_0}{2\cdot 3}x^3 + \frac{a_1}{3\cdot 4}x^4 + \frac{a_0}{2\cdot 3\cdot 5\cdot 6}x^6 + \frac{a_1}{3\cdot 4\cdot 6\cdot 7}x^7 + \ldots$$

Find two independent solutions to this ODE and explain how you know that they're independent.

a) $\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=-2}^{\infty} (n+2)(n+1)a_{n+2}x^n = 0 + 0 + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$

b) Soln is $a_0\left[1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 + \ldots\right] + a_1\left[x + \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 + \ldots\right]$

So two solns. are

$$y_1(x) = 1 + \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 + \ldots$$

& $$y_2(x) = x + \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 + \ldots$$

Use the Wronskian to show they're independent:

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

(8) (15 pts) Find the general solution of the ODE $y^{(4)} - y = 30e^{2t}$.

First, solve the assoc. homogeneous ODE $y^{(4)} - y = 0$.

The char. poly. is $r^4 - 1 = (r^2+1)(r^2-1) = (r^2+1)(r+1)(r-1)$.

Thus the roots are $\pm 1, \pm i$, and the soln. is

$$c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t.$$

Next, we need to find a particular soln. $v(t)$. We guess that $v(t) = ae^{2t}$.

Then $v^{(4)}(t) - v(t) = 16ae^{2t} - ae^{2t} = 30e^{2t}$

$$15ae^{2t} = 30e^{2t}$$

$$a = 2$$

So $v(t) = 2e^{2t}$ is one soln, and the general soln

is $c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t + 2e^{2t}$.

EXTRA CREDIT (2 pts) Which is more fun, an nth-order ODE or a system of n first-order ODEs?