Math 16(1) Practice Second Midterm (not to be turned in)

1. (15 pts.) Compute the determinants of the following matrices.

(a) $\det \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & -2 & 2 \\ 1 & 2 & 2 & -3 \\ 1 & 2 & -1 & 2 \end{bmatrix}$

(b) $\det \begin{bmatrix} -1 & -4 & 2 \\ 0 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$

(c) $\det \begin{bmatrix} a & b & c & d \\ 2a & e+2b & f+2c & g+2d \\ 0 & 0 & 0 & j \\ 0 & 0 & h & i \end{bmatrix}$ [hint: compare with $\det \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$]

2. (10 pts.) Let $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and let $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 4 & 3 \\ 5 & 3 \end{bmatrix}$. Let S be the triangle with vertices at $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$.

(a) Compute the area of S.

(b) Compute the area of the image of S under the map T.

3. (9 pts.) Let A be an $n \times n$ matrix. List three statements that are each equivalent to the statement: "A is invertible."

4. (8 pts.) Suppose A and B are $n \times n$ matrices with

$$\det(A^{-1}) = 2, \ \det(AB) = 3.$$

Find the following determinants.

(a) $\det(A)$

(b) $\det(B)$

(c) $\det(A^2)$

(d) $\det(B^T)$

5. (10 pts.) Let V be a vector space. Complete the following definitions.

(a) $\mathfrak{B} = \{v_1, \ldots, v_n\}$ is a *basis* of V if...

(b) A set H is a *subspace* of V if...

6. (12 pts.) Let $H = \left\{ \begin{bmatrix} a+b-c \\ 3b+3c \\ a-2b-4c \end{bmatrix} : a, b, c \text{ any real numbers} \right\}$.

(a) H is a subspace of $\mathbb{R}^3$. Justify.

(b) What is the dimension of H?

7. (13 pts.) The matrices A and B below are row equivalent.

$$\text{A} = \begin{bmatrix} 1 & 2 & -2 & 0 & 7 \\ -2 & -3 & 1 & -1 & -5 \\ -3 & -4 & 0 & -2 & -3 \\ 3 & 6 & -6 & 5 & 1 \end{bmatrix}, \quad \text{B} = \begin{bmatrix} 1 & 0 & 4 & -1 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for the column space of A. What is the rank of A?

(b) Find a basis for the row space of A.

(c) Find a basis for the null space of A.

8. (15 pts.) Let A be an $m \times n$ matrix. Prove that $\text{Nul}\, A$ is a subspace of $\mathbb{R}^n$.

9. (15 pts.) Let $C[0,1]$ denote the vector space consisting of all continuous functions on $[0,1]$. Let $I : C[0,1] \to \mathbb{R}$ be the transformation given by $I(f) = \int_0^1 f(x)dx$. Is T linear? If so, prove it, if not explain why not.

10. (20 pts.) Let W be the vector space with ordered basis $\{\cos(2x), \sin(2x), 1\}$. Let $D : W \to W$ denote the linear transformation given by $D(f) = f'(x)$.

(a) Find the matrix associated to D with the given basis.
(b) What is the $\dim \ker D$? What is a basis for $\ker D$?
(c) What is $\dim \operatorname{Im} D$? What is a basis for $\operatorname{Im} D$?
(d) Use the matrix representation of D to find $D(3 - 5\cos(2x) + 8\sin(2x))$.

11. (15 pts.) If possible, give an example of each of the following. Your examples should clearly satisfy the given statements, if it's not obvious then you must explain why your example works. If it no such example exists, state 'impossible' and give an explanation of why not.

(a) $\{v_1, v_2, v_3\}$ is a set of linearly dependent vectors with the property that $\operatorname{Span}\{v_1, v_2, v_3\} \neq \operatorname{Span}\{v_2, v_3\}$.
(b) V is a vector space that doesn't have a finite basis.
(c) $T : \mathbb{P}_3 \to \mathbb{P}_2$ is a linear transformation with $\ker T = \{\mathbf{0}\}$.

12. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the map given by $T(p) = tp'' + 2p$. Let $\mathfrak{B} = \{1, t, t^2\}$ be the basis for $\mathbb{P}_2$.

(a) (10 pts.) Show that T is a linear transformation.
(b) (5 pts.) Find A, the matrix for T with respect to the basis $\mathfrak{B}$.
(c) (5 pts.) Find $\operatorname{Nul} A$.
(d) (5 pts.) Find $\operatorname{Col} A$.
(e) (5 pts.) Show that A has only one eigenvalue.
(f) (5 pts.) Find the eigenspace in $\mathbb{R}^3$ corresponding to this eigenvalue.
(g) Use your answers in (a)-(f) to find:
  (i) (4 pts.) $\ker T$
  (ii) (4 pts.) $\operatorname{Im} T$
  (iii) (4 pts.) The eigenspace in $\mathbb{P}_2$ corresponding to the eigenvalue in (e).

13. $C = \begin{bmatrix} -4 & 2 & -10 \\ -5 & 3 & -10 \\ 1 & -1 & 3 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

(a) (9 pts.) Find the corresponding eigenvalues.
(b) (9 pts.) If possible, find a matrix P and a diagonal matrix D so that $C = PDP^{-1}$. If impossible, clearly state why.