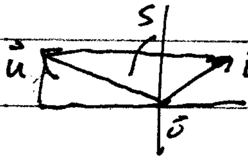
16 Practice 2nd Midterm Solns

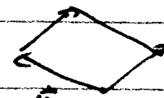
① a) $\det\begin{bmatrix}1&2&3&5\\1&2&-2&2\\1&2&2&-3\\1&2&-1&2\end{bmatrix} = \det\begin{bmatrix}1&2&3&5\\0&0&-5&-3\\0&0&-1&-8\\0&0&-4&-7\end{bmatrix} = 1\cdot\det\begin{bmatrix}0&-5&-3\\0&-1&-8\\0&-4&-7\end{bmatrix} = 0$

b) $\det\begin{bmatrix}-1&-4&2\\0&3&0\\5&2&-1\end{bmatrix} = 3\cdot\det\begin{bmatrix}-1&2\\5&-1\end{bmatrix} = 3(1-10) = -27$

c) $= \det\begin{bmatrix}a&b&c&d\\0&e&f&g\\0&0&h&i\\0&0&0&j\end{bmatrix} = a\cdot e\cdot h\cdot j$

②



a) Area $S = \frac{1}{2}$ Area  $= \frac{1}{2}\left|\det\begin{bmatrix}-2&1\\1&1\end{bmatrix}\right|$

$= \frac{1}{2}|-2-1| = 3/2$

b) $\text{Area}(T(S)) = |\det A|\ \text{Area } S = \left|\det\begin{bmatrix}4&3\\5&3\end{bmatrix}\right|\cdot 3/2 = |12-15|\cdot 3/2 = 9/2$

③ Lots of answers – see Invertible Matrix Theorem

④ a) $\det A = \frac{1}{\det A^{-1}} = \frac{1}{2}$ b) $\det AB = \det A\cdot\det B$, so $3 = \frac{1}{2}\det B$, so $\det B = 6$

c) $\det A^2 = (\det A)^2 = 1/4$ d) $\det B^T = \det B = 6$

⑥ a) $H = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\end{bmatrix}, \begin{bmatrix}-1\\3\\-4\end{bmatrix}\right\}$. Every span is a subspace.

b) $H = \text{Col}\begin{bmatrix}1&1&-1\\0&3&3\\1&-2&-4\end{bmatrix}$. Row reduce to $\begin{bmatrix}1&1&-1\\0&3&3\\0&0&0\end{bmatrix}$. A basis for $H =$ Col space

is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\3\\-2\end{bmatrix}\right\}$, so $\dim H = 2$.

⑦ a) 1st, 2nd, & 4th columns of A: $\left\{\begin{bmatrix}1\\-2\\-3\\3\end{bmatrix}, \begin{bmatrix}2\\-3\\-4\\6\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\\5\end{bmatrix}\right\}$. Rk = 3

b) $1^{st}, 2^{nd}, 3^{rd}$ (nonzero) rows of B: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right\}$

c) Further reduce to reduced echelon form: $\begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & -3 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

so
$$x_1 = -4x_3 + 3x_5$$
$$x_2 = 3x_3 + 5x_5$$
$$x_4 = 4x_5$$
basis is $\left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$

(8) See p. 227.

(9) T is linear: $T(af+bg) = \int_0^1 (af+bg)(x)\,dx = \int_0^1 (af(x)+bg(x))\,dx$

$= a\int_0^1 f(x)\,dx + b\int_0^1 g(x)\,dx = aT(f) + bT(g)$ ✓

(10) a) $D(\vec{b}_1) = D(\cos 2x) = -2\sin 2x = -2\vec{b}_2$

$D(\vec{b}_2) = D(\sin 2x) = 2\cos 2x = 2\vec{b}_1$

$D(\vec{b}_3) = D(1) = 0.$

So $M = \left[D(\vec{b}_1)\ D(\vec{b}_2)\ D(\vec{b}_3) \right] = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) Basis for ker $D = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ - dimension 1

c) dim Im D = rk D. Row reduce to $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Rank is 2, basis is $\left\{ \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

d) $3 - 5\cos 2x + 8\sin 2x = \begin{bmatrix} -5 \\ 8 \\ 3 \end{bmatrix}$. So $D(3 - 5\cos 2x + 8\sin 2x) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 0 \end{bmatrix}$

$= 16\cos 2x + 10\sin 2x$ ✓

(11) a) Lots of possibilities. Here's one: $\left\{\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}0\\1\end{bmatrix}\right\}$

b) Lots of possibilities. Here's one: $\mathbb{P} = \{$ all polynomials $\}$

c) Impossible: $\dim \mathbb{P}_3 = 4$ & $\dim \mathbb{P}_2 = 3$. So ~~that~~ $\dim \operatorname{Im} T \leq 3$,
so $\dim \ker T \geq 4-3 = 1$.

(12) a) $T(ap+bq) = t(ap+bq)'' + 2(ap+bq) = a(tp''+2p) + b(tq''+2q)$
$= aT(p) + bT(q)$

b) $T(\vec{b}_1) = T(1) = t\cdot 0 + 2\cdot 1 = 2 = 2\vec{b}_1$
$T(\vec{b}_2) = T(t) = t\cdot 0 + 2t = 2\vec{b}_2$
$T(\vec{b}_3) = T(t^2) = t\cdot 2 + 2t^2 = 2\vec{b}_2 + 2\vec{b}_3$

So $A = \begin{bmatrix}2&0&0\\0&2&2\\0&0&2\end{bmatrix}$

c) Reduce A to $\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$. $\operatorname{Nul} A = \{\vec{0}\}$

d) $\operatorname{Col} A$ has basis $\left\{\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\2\end{bmatrix}\right\}$ - it's a 3-dim subspace of $\mathbb{P}_2$, so it's all of $\mathbb{P}_2$.

e) $\det(A-\lambda I) = \det\begin{pmatrix}2-\lambda&0&0\\0&2-\lambda&2\\0&0&2-\lambda\end{pmatrix} = (2-\lambda)^3$ - only eigenvalue is 2

f) $= \operatorname{Nul}(A-2I) = \operatorname{Nul}\begin{pmatrix}0&0&0\\0&0&2\\0&0&0\end{pmatrix} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

g) i) $\ker T = 0$ ii) $\operatorname{Im} T =$ all of $\mathbb{P}_2$ iii) $\{a+bt\}$

(13) a) $C\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\-2\\0\end{bmatrix}$, so $\lambda_1 = -2$. $C\cdot\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}6\\6\\-3\end{bmatrix}$, so $\lambda_2 = 3$. $C\cdot\begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}-2\\0\\1\end{bmatrix}$, so $\lambda_3 = 1$.

b) $P = \begin{bmatrix}1&2&-2\\1&2&0\\0&-1&1\end{bmatrix}$, $D = \begin{bmatrix}-2&0&0\\0&3&0\\0&0&1\end{bmatrix}$.