Problem 1

Part A Find the solution set to the following system of equations:

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 15 \\ 4x_1 & + & 5x_2 & + & 6x_3 & + & 7x_4 & = & 6 \\ 6x_1 & + & 7x_2 & + & 8x_3 & + & 9x_4 & = & 0 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 15 \\ 4 & 5 & 6 & 7 & 6 \\ 6 & 7 & 8 & 9 & 0 \end{array}\right] \sim \begin{array}{r} \\ -4R_1+R_2 \\ -6R_1+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 15 \\ 0 & -3 & -6 & -9 & -54 \\ 0 & -5 & -10 & -15 & -90 \end{array}\right]$$

$$\sim \begin{array}{r} \\ -\tfrac{1}{3}R_2 \\ -\tfrac{1}{5}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 15 \\ 0 & 1 & 2 & 3 & 18 \\ 0 & 1 & 2 & 3 & 18 \end{array}\right] \sim \begin{array}{r} -2R_2+R_1 \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -21 \\ 0 & 1 & 2 & 3 & 18 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So X_3, X_4 free $\quad X_1 = -21 + X_3 + 2X_4$

$\quad X_2 = 18 - 2X_3 - 3X_4$

ln set is $\left\{ \begin{bmatrix} -21 + X_3 + 2X_4 \\ 18 - 2X_3 - 3X_4 \\ X_3 \\ X_4 \end{bmatrix} : X_3, X_4 \in \mathbb{R} \right\}$

pts

Part B Consider the system of equations:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -3x_1 & - & x_2 & + & 2x_3 & = & 0 \\ & & 5x_2 & + & 3x_3 & = & 0 \end{array}.$$

Asking if this has a unique solutions is equivalent to which of the following? Provide a list on the following line: 1, 3, 5, 6

1. The vectors $\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ are linearly independent.

2. $\begin{vmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{vmatrix} = 0.$ would need $\neq 0$

3. The equation $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution for every choice of b_1, b_2, b_3.

4. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \text{Span} \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$. This is always true, so is NOT equiv. to above.

5. The transformation $\vec{x} \to \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \vec{x}$ is one-to-one.

6. The rows of $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}$ are linearly independent.

7. The range of the transformation that sends $\vec{x}$ to $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \vec{x}$ is a subset of $\mathcal{R}^3$. Is always true, so NOT equiv.

Part C Find the inverse of the coefficient matrix in part B and use it to solve the matrix equation $\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$$\left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 5 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \underset{-R_2+R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 5 & 3 & 1 & 0 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right] \sim \tfrac{1}{5}R_2 \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3/5 & 1/5 & 0 \\ 0 & 0 & -2 & -3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/5 & -2/5 & 0 \\ 0 & 1 & 1 & 3/5 & 1/5 & 0 \\ 0 & 0 & 1 & 3/2 & 1/2 & -1/2 \end{array}\right] \underset{-R_3+R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 13/10 & 1/10 & -1/2 \\ 0 & 1 & 0 & -9/10 & -3/10 & 1/2 \\ 0 & 0 & 1 & 3/2 & 1/2 & -1/2 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 13/10 & 1/10 & -1/2 \\ -9/10 & -3/10 & 1/2 \\ 3/2 & 1/2 & -1/2 \end{bmatrix}$$

$$A^{-1} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/10 + 1/2 \\ -3/10 - 1/2 \\ 1/2 + 1/2 \end{bmatrix} = \begin{bmatrix} .6 \\ -.8 \\ 1 \end{bmatrix}$$

Problem 2

Part A Is the set of vectors $\begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ -4 \\ 2 \end{bmatrix}$ linearly independent?

Look at $\begin{bmatrix} 0 & -5 & 10 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & 1 & 2 \end{bmatrix}$ and see if we have a pivot in every column

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 0 & -5 & 10 \end{bmatrix} \sim \begin{array}{c} -3R_1+R_2 \\ R_1+R_3 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -10 & -2 \\ 0 & 6 & -2 \\ 0 & -5 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 10 \\ 0 & 6 & -2 \\ 0 & -10 & -2 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ -2R_2+R_4 \\ \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 10 \\ 0 & 6 & -2 \\ 0 & 0 & -22 \end{bmatrix} \sim$$

This is enough to show that there is a pivot in every column.

So Yes, vectors are indep.

Part B Prove the following theorem from our book: Let $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p\}$ be a set of vectors from $\mathcal{R}^n$. If $p > n$, then this set is linearly dependent.

Let $A = [\vec{v}_1 \cdots \vec{v}_p]$ So A is $n \times p$ matrix.

The most pivots A can have is if there is 1 in every row (which is n.), which is < # of columns

So we can't have a pivot in every column, thus set must be dependent.

9 pts

Problem 3

Part A Let $T : \mathcal{R}^2 \to \mathcal{R}^2$ be the transformation that multiplies a vector by the scalor 2 and then reflects it across the x-axis. Prove that this is a linear transformation.

So what does T do to $\begin{bmatrix} x \\ y \end{bmatrix}$? $\rightsquigarrow \begin{bmatrix} 2x \\ 2y \end{bmatrix} \to \begin{bmatrix} 2x \\ -2y \end{bmatrix}$

So $T\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} x+a \\ y+b \end{bmatrix}\right) = \begin{bmatrix} 2(x+a) \\ -2(y+b) \end{bmatrix} = \begin{bmatrix} 2x \\ -2y \end{bmatrix} + \begin{bmatrix} 2a \\ -2b \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix} + T\begin{bmatrix} a \\ b \end{bmatrix}$

and $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} 2cx \\ -2cy \end{bmatrix} = c\begin{bmatrix} 2x \\ -2y \end{bmatrix} = c\,T\begin{bmatrix} x \\ y \end{bmatrix}$.

So T is linear

7 pts

Part B Now that you know T is linear, you know it has a standard matrix. Find it.

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

So standard matrix for T is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

Part C Is T one-to-one and onto? Justify your answer, of course.

Since A is a square matrix with a pivot in every row & column, it is 1-1 and onto.

or

Use IMT: Since $\det A \neq 0$, A invertible

then transf is 1-1 & onto

Part D Let S be the unit circle in $\mathcal{R}^2$. What is the area of $T(S)$?

area of $T(S)$ = $|\det A| \cdot$ area of S

$= 4 \quad \pi(1)^2$

$= 4\pi$.

Problem 4

Part A Let A be an $n \times n$ matrix whose columns are independent. Show that the columns of A^2 span $\mathcal{R}^n$.

~~Col. of A are indep $\Rightarrow$ det $A \neq 0$ by IMT.~~

~~$\det(A^2) = (\det A)(\det A)$ [rule of det]~~

~~$\neq 0$ by above.~~

~~So by IMT, col. of (A^2) span $\mathbb{R}^n$~~

columns ind. $\Rightarrow$ T_A 1-1 $\Rightarrow$ A inveatble $\Rightarrow$ $A^2 = A \cdot A$ is invertible $\Rightarrow$ T_{A^2} is onto $\Rightarrow$ columns of A^2 span

Part B How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

Need 5 so there will be a pivot in every column.

Part C Find the value of $\begin{vmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{vmatrix}$.

$$-2\begin{vmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{vmatrix} = -6\begin{vmatrix} 4 & 3 & -5 \\ 5 & 2 & -3 \\ 0 & -1 & 2 \end{vmatrix} = -6\left[4(4-3) - 5(6-5)\right]$$

$$= -6\left[4(1) - 5(1)\right] = +6.$$

ach

Problem 5 Mark the following True or False.

T • If the system $A\vec{x} = \vec{b}$ has more than one solution, then so does the system $A\vec{x} = \vec{0}$. Must have free variable

T • If A is a 6×5 matrix, the linear transformation that sends $\vec{x}$ to $A\vec{x}$ cannot map $\mathcal{R}^5$ onto $\mathcal{R}^6$. because have row without pivot

T • If A and B are $m \times n$, then both AB^T and $A^T B$ are defined.

$m \times n \quad n \times m \;,\quad n \times m \quad m \times n$

F • If A and B are $n \times n$ then $(A + B)(A - B) = A^2 - B^2$.

$AA - AB + BA - BB$ Don't have $AB = BA$ usually

F • If A and B are $n \times n$ with det $A = 2$ and det $B = 3$ then det $(A + B) =$ 5.

$\det(A+B) \neq \det A + \det B$

F • If A is invertible then det $(A^{-1}) =$ det A.

$\det A^{-1} = \frac{1}{\det A}$