16 Practice Final Solns

① a) Row reduce the augmented matrix: $\left[\begin{array}{cc|c} 1 & -6 & -1 \\ 1 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 7 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -6 & -1 \\ 0 & 4 & 3 \\ 0 & 7 & 2 \\ 0 & 13 & 7 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} 1 & -6 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -13/4 \\ 0 & 0 & -11/4 \end{array}\right]$: Get $0 = -13/4$ & $0 = -11/4$ – there are no solns.

b) Solve $M^T M \hat{x} = M^T \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \hat{x} = \begin{bmatrix} 8 \\ 45 \end{bmatrix}$, or $\hat{x} = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$.

c) The error is $\| M\hat{x} - \vec{b} \| = \left\| \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\|$

$= \left\| \begin{bmatrix} -1 \\ 1 \\ \cancel{3/2}\ 5/2 \\ -3/2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ -1 \\ 3/2 \\ -3/2 \end{bmatrix} \right\| = \sqrt{0 + 1 + 9/4 + 9/4} = \sqrt{11/2}$.

② a) The only soln. to $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$ is clearly $c_1 = c_2 = c_3 = 0$.

b) Use Gram-Schmidt: $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \dfrac{2 + 2 + 1 + 0}{2 + 1 + 1 + 0} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix}$

$\vec{v}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \dfrac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix}}{\begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix}} \begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \dfrac{2 + 1 + 1 + 0}{4} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \dfrac{-\frac{1}{2} + 3/4 - 1/4 + 0}{\quad} \begin{bmatrix} -1/4 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

c) $\widehat{\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix}} = \dfrac{\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix} \cdot \vec{v}_1}{\|\vec{v}_1\|} \vec{v}_1 + \dfrac{\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix} \cdot \vec{v}_2}{\|\vec{v}_2\|} \vec{v}_2 + \dfrac{\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix} \cdot \vec{v}_3}{\|\vec{v}_3\|} \vec{v}_3 = \dfrac{24}{4} \vec{v}_1 + \dfrac{3}{3/4} \vec{v}_2 + \dfrac{2}{1} \vec{v}_3$

$= \begin{bmatrix} 5 & 4 \\ 0 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix} - \widehat{\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix}} = \begin{bmatrix} -4 & 4 \\ 12 & 0 \end{bmatrix}$ is in $W^\perp$.

③ a) Need $T(ap(t)+bq(t)) = aT(p(t)) + bT(q(t))$ — this follows from the properties of antidifferentiation.

b) If the basis for $\mathbb{P}_2$ is $\vec{u}_1 = t^2, \vec{u}_2 = t, \vec{u}_3 = 1$, & the basis $\mathcal{B}$ for $\mathbb{P}_3$ is $\vec{v}_1 = t^3, \vec{v}_2 = t^2, \vec{v}_3 = t, \vec{v}_4 = 1$, then the matrix is

$$\left[[T(\vec{u}_1)]_{\mathcal{B}} \; [T(\vec{u}_2)]_{\mathcal{B}} \; [T(\vec{u}_3)]_{\mathcal{B}} \right] = \left[\left[\tfrac{1}{3}t^3\right]_{\mathcal{B}} \; \left[\tfrac{1}{2}t^2\right]_{\mathcal{B}} \; [t]_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

c) Example: $T(1+t+t^2) = t + \frac{t^2}{2} + \frac{t^3}{3}$, or, in matrix notation,

$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} = t^3/3 + t^2/2 + t$$

④ a) ii, iii, v, vii, viii

b) Write $\vec{v} = c_1\vec{u}_1 + \dots + c_m\vec{u}_m$. Then $\vec{v} \cdot \vec{u}_i = c_i$, since the $\vec{u}_i$'s are orthonormal. Also, $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = c_1^2 + \dots + c_m^2$, again because the $\vec{u}_i$'s are orthonormal. So $\|\vec{v}\|^2 = |\vec{v}\cdot\vec{u}_1|^2 + \dots + |\vec{v}\cdot\vec{u}_m|^2$.

⑤ a) They're linearly independent, since neither is a multiple of the other, & they of course span their own span, so they're a basis

b) $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, so they're in W. They're linearly independent, and $\dim W = 2$, so they form a basis.

c) $P_{\mathcal{B}\leftarrow A} = \left[\begin{bmatrix}1\\2\\1\end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix}2\\3\\1\end{bmatrix}_{\mathcal{B}} \right] =$

Need to write $\begin{bmatrix}1\\2\\1\end{bmatrix} = c_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}4\\7\\3\end{bmatrix}$, & $\begin{bmatrix}2\\3\\1\end{bmatrix} = d_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + d_2\begin{bmatrix}4\\7\\3\end{bmatrix}$.

Combine into 1 augmented matrix $\left[\begin{array}{cc|cc} -1 & 4 & 1 & 2 \\ 0 & 7 & 2 & 3 \\ 1 & 3 & 1 & 1 \end{array}\right]$, & row reduce:

$$\left[\begin{array}{cc|cc} -1 & 4 & 1 & 2 \\ 0 & 7 & 2 & 3 \\ 0 & 7 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 4 & 1 & 2 \\ 0 & 1 & 2/7 & 3/7 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|cc} -1 & 0 & -1/7 & 2/7 \\ 0 & 1 & 2/7 & 3/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 1/7 & -2/7 \\ 0 & 1 & 2/7 & 3/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$ So $c_1 = 1/7$, $c_2 = 2/7$, $d_1 = -2/7$, & $d_2 = 3/7$,

ie, $\begin{bmatrix}1\\2\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}1/7\\2/7\end{bmatrix}$ & $\begin{bmatrix}2\\3\\1\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}-2/7\\3/7\end{bmatrix}$, so $P_{\mathcal{B}\leftarrow A} = \begin{bmatrix}1/7 & -2/7\\2/7 & 3/7\end{bmatrix}$

d) $\begin{bmatrix}3\\5\\2\end{bmatrix}_A = \left(\begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}2\\3\\1\end{bmatrix}\right)_A = \begin{bmatrix}1\\1\end{bmatrix}$. $\begin{bmatrix}3\\5\\2\end{bmatrix}_{\mathcal{B}} = \left(-\frac{1}{7}\begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{5}{7}\begin{bmatrix}4\\7\\3\end{bmatrix}\right)_{\mathcal{B}} = \begin{bmatrix}-\frac{1}{7}\\ \frac{5}{7}\end{bmatrix}$

(solve by inspection, or by solving $\begin{bmatrix}-1 & 4\\0 & 7\\1 & 3\end{bmatrix}\vec{x} = \begin{bmatrix}3\\5\\2\end{bmatrix}$).

(6 comes later) Note that $\begin{bmatrix}-\frac{1}{7}\\ \frac{5}{7}\end{bmatrix} = \begin{bmatrix}1/7 & -2/7\\2/7 & 3/7\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$, or $\begin{bmatrix}3\\5\\2\end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}\leftarrow A}\begin{bmatrix}3\\5\\2\end{bmatrix}_A$.

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a) $$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{2} & 1 & -\frac{3}{2} \\ 0 & 1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 & \frac{1}{2} \end{array}\right]$$, so the inverse is $\begin{bmatrix}\frac{9}{2} & 1 & -3/2\\ 4 & 0 & -1\\ -\frac{3}{2} & 0 & \frac{1}{2}\end{bmatrix}$, and

$$\vec{x} = \begin{bmatrix}\frac{9}{2} & 1 & -3/2\\ 4 & 0 & -1\\ -\frac{3}{2} & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}8\\6\\-2\end{bmatrix}$$

b) Following the row reduction above, the det. is equal to

$$-\det\begin{bmatrix}1&0&3\\0&1&2\\0&3&8\end{bmatrix} = -\det\begin{bmatrix}1&0&3\\0&1&2\\0&0&2\end{bmatrix} = -2$$

c) Just check: $AB\cdot(B^{-1}A^{-1}) = A(BB^{-1})A = AIA^{-1} = AA^{-1} = I$,
so $B^{-1}A^{-1} = (AB)^{-1}$.

6 a) Char. poly. is $\det\begin{bmatrix}4-\lambda&0&-2\\2&5-\lambda&4\\0&0&5-\lambda\end{bmatrix} = (5-\lambda)\det\begin{bmatrix}4-\lambda&0\\2&5-\lambda\end{bmatrix}$

$= (5-\lambda)(4-\lambda)(5-\lambda)$, so the eigenvalues are 4 & 5.

E_4 = null space of $A-4I = \begin{bmatrix}0&0&-2\\2&1&4\\0&0&1\end{bmatrix} \to \begin{bmatrix}2&1&4\\0&0&1\\0&0&0\end{bmatrix} \to \begin{bmatrix}2&1&0\\0&0&1\\0&0&0\end{bmatrix} = \text{Span}\left\{\begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$

E_5 = null space of $A-5I = \begin{bmatrix}-1&0&-2\\2&0&4\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&2\\0&0&0\\0&0&0\end{bmatrix} = \text{Span}\left\{\begin{bmatrix}-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

b) A is diagonalizable, since $\mathbb{R}^3$ has a basis of eigenvectors for A.

c) The entries of D are the eigenvalues of B.

d) B is invertible $\Rightarrow \det B \neq 0 \Rightarrow$ (product of eigenvalues) $\neq 0$
$\Rightarrow$ no eigenvalue is 0. (There are other ways to prove this.)