FINAL EXAM

MATH 16
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1. (a) Show that the following system of equations is inconsistent:
$$\begin{array}{rcrcr} x & - & 6y & = & -1 \\ x & - & 2y & = & 2 \\ x & + & y & = & 1 \\ x & + & 7y & = & 6 \end{array}$$

   (b) Find the least squares solution for the above equation.

   (c) Find the least-squares error associated to the solution you found above.

2. The following rule gives an inner product on M_2, the set of all 2×2 matrices:
$$\left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right\rangle =$$
$$2a_{11}b_{11} + (a_{11} + a_{12})(b_{11} + b_{12}) + (a_{11} + a_{21})(b_{11} + b_{21}) + a_{22}b_{22}.$$

   (a) Show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a linearly independent set from M_2.

   (b) Let $W = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Find an orthogonal basis for W, using the above inner product.

   (c) Find the projection of $\begin{bmatrix} 1 & 8 \\ 12 & 2 \end{bmatrix}$ onto W.

   (d) Find a nontrivial matrix in $W^{\perp}$

3. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be defined as integration, with constant of integration zero. For example, $T(1 + t + t^2) = t + \frac{t^2}{2} + \frac{t^3}{3}$.
   (a) Show this is a linear transformation.

   (b) Find the matrix representation of T relative to the standard bases for $\mathcal{P}_2$ and $\mathcal{P}_3$.

   (c) Give an example of how this matrix relates to the transformation T.

4. (a) Let A be a 12×12 matrix. You are interested in showing that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathcal{R}^{12}$. This is equivalent to which of the following? Write the appropriate numbers here:

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   (i) det $A = 0$.
   (ii) the transformation $\vec{x} \to A\vec{x}$ is one-to-one.
   (iii) Nul $A = \{\vec{0}\}$.
   (iv) A is diagonalizable.
   (v) rank $A = 12$.
   (vi) $A = A^T$.
   (vii) the columns of A span $\mathcal{R}^{12}$.
   (viii) A is invertible.

(b) Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ is an orthonormal list of vectors in vector space V. Prove that if $\vec{v} \in \text{Span}\,\{u_1, \cdots, u_m\}$ then
$||\vec{v}||^2 = |<\vec{v}, \vec{u}_1>|^2 + \cdots + |<\vec{v}, \vec{u}_m>|^2$.

5. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$.

(a) Show that $\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

(b) Show that $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} \right\}$ is another basis for W.

(c) Find the change of basis matrix from $\mathcal{A}$ to $\mathcal{B}$, i.e. $P_{\mathcal{B} \leftarrow \mathcal{A}}$.

(d) Find the coordinate vector of $\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ in both bases, and use your answer from the last question to relate them.

6. (a) Find the eigenvalues of $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, and find the eigenspace associated to each eigenvalue.

(b) Is matrix A diagonalizable?

(c) Let B be an arbitrary square matrix. If B is diagonalizable, what does this diagonal matrix represent?

(d) Prove that if a matrix B is invertible then 0 is not an eigenvalue of B.

7. (a) Find the inverse to $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix}$ and use it to solve

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

(b) Find the determinent of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 3 & 8 \end{bmatrix}$.

(c) Prove that the inverse of AB is $B^{-1}A^{-1}$.