1. (20 pts) Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$. Find M^{-1}, and use it to solve the equation $M\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$M \qquad I$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$I \qquad M^{-1}$

$$M^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$M\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \text{so} \quad \vec{x} = M^{-1}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

2. (20 pts) Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false. (If it's true, you don't have to say anything else.)

(a) The transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is linear.

FALSE - $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, but $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for any linear transformation

(b) If M is an invertible matrix, then $(M^2)^{-1} = (M^{-1})^2$.

TRUE

(c) The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

FALSE - $\det\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = 1 \cdot 6 - 3 \cdot 2 = 0$

(d) There is an invertible matrix M such that $M^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

FALSE - If M is invertible, then so is M^{-1}, but $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible ($\det\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$)

(e) There is a matrix M such that $M\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

TRUE

(f) If M is a 2×3 matrix, then the equation $M\mathbf{x} = \mathbf{b}$ has either zero or infinitely many solutions.

TRUE

3. (20 pts) The color of light can be represented in a vector $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$, where R is the amount of red, G is the amount of green, and B is the amount of blue. The human eye and the brain transform the incoming signal into the signal $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$, where

$$\text{intensity} \quad I = \frac{R+G+B}{3}$$

$$\text{long-wave signal} \quad L = R - G$$

$$\text{short-wave signal} \quad S = B - \frac{R+G}{2}.$$

Find the matrix M representing the transformation from $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ to $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$.

$$M = \left[M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; M\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

Here, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \frac{1+0+0}{3} \\ 1-0 \\ 0 - \frac{1+0}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{-1}{2} \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \frac{0+1+0}{3} \\ 0-1 \\ 0 - \frac{0+1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -1 \\ \frac{-1}{2} \end{bmatrix}$,

and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \frac{0+0+1}{3} \\ 0-0 \\ 1 - \frac{0+0}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$. So $M = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 1 \end{bmatrix}$

4. (20 pts) The Springfield Zoo has two kinds of animals, monkeys and lions. One monkey eats $\$h_M$ worth of hamburgers in a month, and uses $\$f_M$ worth of fur-care products. Similarly, each lion eats $\$h_L$ worth of hamburgers in a month, and uses $\$f_L$ worth of fur-care products.

(a) If the zoo has n_M monkeys and n_L lions, how do you interpret the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h_M & h_L \\ f_M & f_L \end{bmatrix} \begin{bmatrix} n_M \\ n_L \end{bmatrix}$?

(b) How do you interpret the vector $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} h_M & h_L \\ f_M & f_L \end{bmatrix} \begin{bmatrix} n_M \\ n_L \end{bmatrix}$?

a) $\begin{bmatrix} h_m & h_L \\ f_m & f_L \end{bmatrix} \begin{bmatrix} n_m \\ n_L \end{bmatrix} = \begin{bmatrix} h_m n_M + h_L n_L \\ f_m n_M + f_L n_L \end{bmatrix} = \begin{bmatrix} \text{\$ spent on hamburgers in a month} \\ \text{\$ spent on fur-care products in a month} \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} h_m & h_L \\ f_m & f_L \end{bmatrix} \begin{bmatrix} n_M \\ n_L \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} h_m n_M + h_L n_L \\ f_M n_M + f_L n_L \end{bmatrix}$

$= \begin{bmatrix} h_m n_m + h_L n_L + f_m n_m + f_L n_L \end{bmatrix}$

$=$ (\$ spent on hamburgers + \$ spent on fur-care products) per month

5. (20 pts) Show that if the columns of a matrix M are linearly dependent, then the transformation defined by multiplication by M is not one-to-one.

Say $M = [\vec{v}_1 \cdots \vec{v}_n]$. Since the columns are lin. dep., there are weights $c_1, \ldots, c_n$ (at least one of which is nonzero) such that

$c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n = \vec{0}$.

We can rewrite this eqn. as $M \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$.

~~So~~ Therefore $M \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ & $M \cdot \vec{0}$ both equal $\vec{0}$.

Since $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq \vec{0}$, mult. by M is not 1-1.

(EXTRA CREDIT) Which are better, matrices or linear transformations? Justify your answer.