1. Let A and B be 4×4 matrices with $\det(A) = 5$ and $\det(B) = 7$. Find the following, or state that you do not have enough information.

(a) $\det(A^{-1})$

$= \frac{1}{\det A} = \frac{1}{5}$

(b) $\det(AB) = (\det A)(\det B) = 35$

(c) $\det(A + A)$ $= \det(2A)$. Each row gets multiplied by 2, so

$\det(2A) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot \det A = 80$

(d) $\det(A + B)$ Not enough info.

(e) $\det(A^T)$ $= \det A = 5$

(f) $\det(A^2)$ $= (\det A)^2 = 25$

2. Orthogonally diagonalize the matrix $M = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. (HINT: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 6, and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are in the null space of M.)

First, find an orthogonal basis of eigenvectors. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is already $\perp$ to $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, since its eigenvalue is 1 & theirs is 0. So apply Gram-Schmidt to $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis of eigenvectors. Next, normalize to get an orthonormal basis: $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$.

Then $M = PDP^{-1} = PDP^T$, where

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \quad \& \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

3. State whether each of the following is true or false. If it's false, explain why or give an example showing that it's false. (If it's true, you don't need to say anything more.)

(a) The set $\{\cos^2 t, \sin^2 t, 1\}$ is a linearly independent subset of the space of all continuous functions.

False : $\cos^2 t + \sin^2 t - 1 = 0$

(b) The matrix $\begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & -3 \\ 5 & -3 & 1 \end{bmatrix}$ has $\sqrt{2} + \sqrt{2}i$ as an eigenvalue.

False - the matrix is symmetric, so all of its eigenvalues are real

(c) If M is a square matrix with only 0 as an eigenvalue, then M is the zero matrix.

False. Ex: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) Let $\mathbb{P}_2$ be the space of all polynomials of degree ≤ 2. Then the transformation $T : \mathbb{P}_2 \to \mathbb{R}$ given by $T(p(t)) = p(7)$ is linear.

True

(e) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of vectors in the vector space V. The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a subspace of V.

True

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(f) It is possible to find an 8×7 matrix M and a vector $\mathbf{b} \in \mathbb{R}^8$ such that the equation $M\mathbf{x} = \mathbf{b}$ has exactly one solution.

True

(g) Let M be a 6×4 matrix with $\dim \operatorname{Nul} M^T = 4$. Then it is possible to find a vector $\mathbf{b} \in \mathbb{R}^6$ such that the equation $M\mathbf{x} = \mathbf{b}$ has exactly one solution.

False - $\dim \operatorname{Nul} M^T + \dim \operatorname{Col} M^T = 6$, so $\dim \operatorname{Col} M^T = 2$.
$\operatorname{Col} M^T = \operatorname{Row} M$, so the rank of M is 2, and $\dim \operatorname{Nul} M = 4-2=2$.
So $M\vec{x} = \vec{b}$ has either no solutions or infinitely many solns.

(h) Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Then the system necessarily has a solution for every possible choice of constants on the right sides of the equations.

True

(i) If M is an $m \times n$ matrix with orthonormal columns, then $M\mathbf{x} \cdot M\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$.

True

(j) The vector $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for the matrix $\begin{bmatrix} -1 & 4 & 2 \\ -1 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$.

True

4. Find bases for the row space, column space, and null space of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix}$. (Be sure that I can tell which is which.)

Row reduce: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A basis for the row space is the set of nonzero rows of the <u>reduced</u> matrix, ie, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for the column space is the set of pivot columns of the <u>original</u> matrix, ie, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix} \right\}$.

A basis for the null space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Fill in the blanks with the letters of all the appropriate properties, or write "NONE." (You don't need to write that every symmetric matrix is symmetric, for example.)

A. symmetric
B. orthogonally diagonalizable
C. invertible
D. one-to-one
E. square
F. diagonal
G. orthogonal

I. A symmetric matrix must also be: B, E

II. An orthogonally diagonalizable matrix must also be: A, E

III. An invertible matrix must also be: D, E

IV. A one-to-one matrix must also be: NONE

V. A square matrix must also be: NONE

VI. A diagonal matrix must also be: A, B, E

VII. An orthogonal matrix must also be: C, D, E

VIII. A diagonalizable matrix must also be: E

Let $\mathbb{P}$ be the vector space of all polynomials, and let $T : \mathbb{P} \to \mathbb{P}$ be the transformation given by $T(p(t)) = tp'(t)$. (For example, $T(t^2+1) = t(2t) = 2t^2$.) Find the eigenvectors and eigenvalues for T. (HINT: Remember that $\mathbb{P}$ is infinite dimensional.)

$T(a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0) = n a_n t^n + (n-1) a_{n-1} t^{n-1} + \dots + 2 a_2 t^2 + a_1 t$.

So if $p(t) = a_n t^n + \dots + a_0$ is an eigenvector with eigenvalue d, then

$n a_n t^n + (n-1) a_{n-1} t^{n-1} + \dots + 2 a_2 t^2 + a_1 t = d a_n t^n + d a_{n-1} t^{n-1} + \dots + d a_1 t + d a_0$

This can happen only if all but one of the coefficients a_i is 0.

Then $T(a_n t^n) = n a_n t^n$. So, for each $n = 0, 1, 2, \dots$,

t^n is an eigenvector with eigenvalue n.

If possible, give an example of each of the following. Your examples should clearly satisfy the given statements – if it's not obvious then you must explain why your example works. If no such example exists, state "IMPOSSIBLE" and explain.

(a) M is a 3×3, nonzero matrix with $M^2 = 0$, the zero matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (there are other answers)

(b) M is a 3×3 matrix with no eigenvectors in $\mathbb{R}^3$.

Impossible – a 3x3 matrix must have at least one real eigenvalue (since ~~its~~ its characteristic polynomial is a cubic), and thus at least one real eigenvector.

(c) M is a matrix that is one-to-one but not onto.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ (there are other answers)

(d) M is a matrix that is onto but not one-to-one.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (there are other answers)

8 Find the quadratic function $y = c_0 + c_1 x + c_2 x^2$ that best fits the data points $(-1, 8), (0, 8), (1, 4)$, and $(2, 16)$.

Find the least-squares soln to

$$\begin{aligned} c_0 - c_1 + c_2 &= 8 \\ c_0 &= 8 \\ c_0 + c_1 + c_2 &= 4 \\ c_0 + 2c_1 + 4c_2 &= 16 \end{aligned}, \text{ or } \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{M} \vec{c} = \underbrace{\begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}}_{\vec{b}}$$

Solve $M^T M \hat{c} = M^T \vec{b}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \hat{c} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \vec{c} = \begin{bmatrix} 36 \\ 28 \\ 76 \end{bmatrix}$$

Row reduce

$$\left[\begin{array}{ccc|c} 4 & 2 & 6 & 36 \\ 2 & 6 & 8 & 28 \\ 6 & 8 & 18 & 76 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 1 & 3 & 18 \\ 1 & 3 & 4 & 14 \\ 3 & 4 & 9 & 38 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 2 & 1 & 3 & 18 \\ 3 & 4 & 9 & 38 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & -5 & -5 & -10 \\ 0 & -5 & -3 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & 1 & 1 & 2 \\ 0 & -5 & -3 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 14 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

So $\hat{c} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$, and the best-fitting quadratic is $5 - x + 3x^2$.

8. I won't give any partial credit on this problem, so check your answers.

Let $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

(a) Find M^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

So $M^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$

(b) Use your answer from part (a) to help you solve the equation

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M\vec{x} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix},$$

$$M\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{x} = M^{-1}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 49 \\ -9 \\ -36 \end{bmatrix}$$

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~~9~~. Recall that $M_{2\times 2}$ is the vector space of all 2×2 matrices. Let $T:\mathbb{R}^3 \to M_{2\times 2}$ be the transformation given by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

(a) Show that T is linear.

$$T\left(c_1\begin{bmatrix} a \\ b \\ c \end{bmatrix} + c_2\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) = T\left(\begin{bmatrix} c_1 a + c_2 d \\ c_1 b + c_2 e \\ c_1 c + c_2 f \end{bmatrix}\right) = \begin{bmatrix} c_1 a + c_2 d & c_1 b + c_2 e \\ c_1 b + c_2 e & c_1 a + c_2 d \end{bmatrix}$$

$$= c_1\begin{bmatrix} a & b \\ b & a \end{bmatrix} + c_2\begin{bmatrix} d & e \\ e & d \end{bmatrix} = c_1 T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + c_2 T\left(\begin{bmatrix} d \\ e \\ f \end{bmatrix}\right) \checkmark$$

(b) What is the kernel of T?

If $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a = b = 0$. So $\ker T = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

(c) What is the image (or range) of T?

$\operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$

(d) Find the matrix M for T, under the standard basis for $\mathbb{R}^3$ and the basis

$$\mathcal{B} = \left\{\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

for $M_{2\times 2}$.

$$M = \begin{bmatrix} \left(T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathcal{B}} & \left(T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)_{\mathcal{B}} & \left(T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{\mathcal{B}} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{B}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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(e) Find the matrix N for T, under the standard basis for $\mathbb{R}^3$ and the basis

$$\mathcal{C} = \left\{ \mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

for $M_{2\times 2}$.

$$N = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_e \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_e \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_e \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) Find a matrix P such that $M = PNP^{-1}$.

P is the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$

$$P = \left[(\vec{u}_1)_{\mathcal{B}} \quad (\vec{u}_2)_{\mathcal{B}} \quad (\vec{u}_3)_{\mathcal{B}} \quad (\vec{u}_4)_{\mathcal{B}} \right]$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$