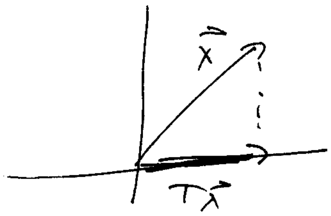
1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects vectors onto the x-axis. What are the eigenvectors and corresponding eigenvalues for T?



Any vector lying along the x-axis is sent to itself - it's an eigenvector with eigenvalue 1.

Any vector lying along the y-axis is sent to $\vec{0}$ - it's an eigenvector with eigenvalue 0.

2. Let A and B be 4×4 matrices with $\det(A) = 4$ and $\det(B) = 6$. Find the following, or state that you do not have enough information.

(a) $\det(A^{-1})$ $= \dfrac{1}{\det A} = \dfrac{1}{4}$

(b) $\det(AB)$ $= \det A \cdot \det B = 24$

(c) $\det(A + A)$ $= \det(2A)$. Each row is multiplied by 2, so

$\det(2A) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot \det A = 16 \cdot 4 = 64$

(d) $\det(A + B)$ not enough info

(e) $\det(A^T)$ $= \det A = 4$

(f) $\det(A^2)$ $= \det A \cdot \det A = 16$

3. Let A and B be 2×2 matrices. Let the eigenvalues of A be 2 and -3, and the eigenvalues of B 4 and 5. Find the following, or state that you do not have enough information.

(a) The eigenvalues of A^{-1}. $\frac{1}{2}$ & $-\frac{1}{3}$ – the inverse of the eigenvalues of A

(b) The eigenvalues of AB. Not enough info

(c) The eigenvalues of $A + A$. $= 2A$, so $2\times$ the eigenvalues of A,
so 4 & -6

(d) The eigenvalues of $A + B$. Not enough info.

(e) The eigenvalues of A^T. = eigenvalues of $A = 2, -3$

(f) The eigenvalues of A^2. = squares of eigenvalues of A = 4 & 9

4. Let $M = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If possible, diagonalize M (that is, find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$). If it's not possible, prove it.

Need a basis of eigenvectors. First, need the eigenvalues. Since M is upper triangular, they're just the diagonal entries, $2, 0, 0$.

Now, find the corresponding eigenvectors:

<u>2-eigenspace</u> = nullspace of $M - 2I = \begin{bmatrix} 0 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Row reduce to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ - basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

<u>0-eigenspace</u> = nullspace of $M - 0I = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$x = -y - z$, so basis is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

So: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is one answer.

5. Decide whether each of the following statements is true or false. If one is false, explain why or give an example showing that it's false. (If it's true, you don't have to say anything else.)

(a) If M is similar to N, and N is diagonalizable, then M is diagonalizable.

TRUE ($M = QNQ^{-1}$, & $N = PDP^{-1}$, so plug in to get
$$M = QPDP^{-1}Q^{-1} = (QP)D(QP)^{-1}.)$$

(b) The vector $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ is in the left null space of the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

TRUE $\left([1\ -2\ 1]\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = [0\ 0]. \right)$

(c) The row space of a matrix M has the same dimension as the column space of M.

TRUE (count the pivots)

(d) A square matrix is onto if and only if 0 is not an eigenvalue.

TRUE (0 is not an eigenvalue $\Leftrightarrow$ $\det M \neq 0$ $\Leftrightarrow$ M is invertible $\Leftrightarrow$ M is onto)

(e) If M is a 6×4 matrix, then the dimension of Nul M is at least $6 - 4 = 2$.

FALSE – Rank M could be 4, so then $\dim \operatorname{Nul} M = 4 - \operatorname{Rank} M = 0$.

(f) The transformation $T : M_{2\times 2} \to \mathbb{P}_2$ that sends each 2×2 matrix to its characteristic polynomial is linear.

FALSE: $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \lambda I\right)$
$T(\vec{0})$ =
$= \det\left(\begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2 \neq \vec{0}$.

6. The vector space $M_{2\times 2}$ of 2×2 matrices has basis

$$\mathcal{B} = \left\{ \mathbf{x}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Find the matrix for the linear transformation $T(M) = M^T - 2M$ with respect to this basis.

Matrix is $\left[[T\vec{x}_1]_{\mathcal{B}} \cdots [T(\vec{x}_4)]_{\mathcal{B}} \right]$

$$T(\vec{x}_1) = T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = -\vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(\vec{x}_2) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} = -2\vec{x}_2 + \vec{x}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{x}_3) = T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \vec{x}_2 - 2\vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}$$

$$T(\vec{x}_4) = T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = -\vec{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

So the matrix is $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(EXTRA CREDIT) Let M be a 2×2 matrix with eigenvalues 0 and 1. Show that $M^2\mathbf{x} = M\mathbf{x}$ for every vector $\mathbf{x}$. Let $\vec{x}_1$ & $\vec{x}_2$ be eigenvectors w/ evalues 0 & 1, respectively.

Then they form a basis for $\mathbb{R}^2$. Write $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$.

Then $M\vec{x} = 0 \cdot c_1\vec{x}_1 + 1 \cdot c_2\vec{x}_2 = c_2\vec{x}_2$, & $M^2\vec{x} = M(M\vec{x}) = c_2\vec{x}_2$.