∇

gradient, divergence and curl

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Last time . . .

- Differential calculus of functions $f : \mathbf{R}^m \to \mathbf{R}$, $m = 2,\ 3$
- Directional derivatives $D_{\hat{\mathbf{n}}} f$ given in terms of *vector gradient* ∇f:

$$D_{\hat{\mathbf{n}}} f = \nabla f \cdot \hat{\mathbf{n}}$$

where

$$\nabla f := \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{pmatrix} \qquad \text{for } m = 3.$$

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New questions:

- What is ∇ ?
- Can it be used in other ways?

The vector ∇ (pronounced *Del*)

Vector gradient

$$\nabla f := \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} f$$

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Vector gradient

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Therefore might say

$$\nabla := \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Can ∇ be used in other ways?

Divergence $= \nabla\cdot$

Suppose given *vector field* $\mathbf{u} : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$.

$\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\mathbf{u} = (u, v, w)^{\mathrm{T}}$, $\mathbf{x} = (x, y, z)^{\mathrm{T}}$.

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Define the divergence $\nabla \cdot \mathbf{u}$ (pronounced *div* $\mathbf{u}$)

$$\nabla \cdot \mathbf{u} := \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

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$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

NB Similar definitions for $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ for $n \neq 3$.

Curl $= \nabla \times$

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Define $\nabla \times \mathbf{u}$ (pronounced *curl* $\mathbf{u}$)

$$\nabla \times \mathbf{u} := \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{pmatrix}$$

$$= \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix}.$$

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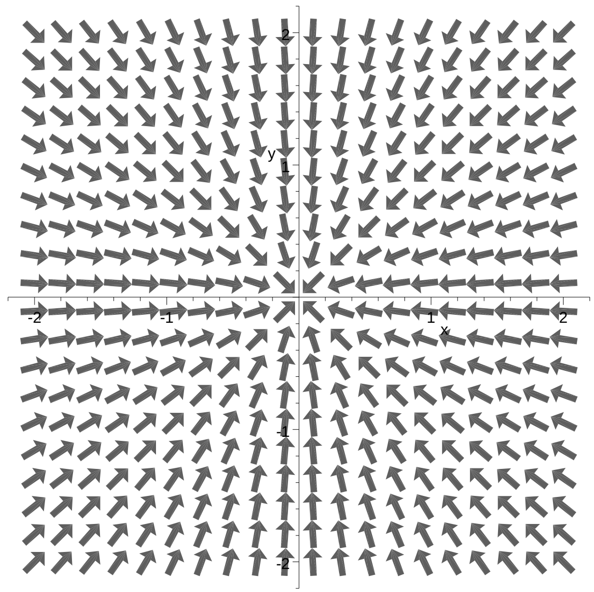
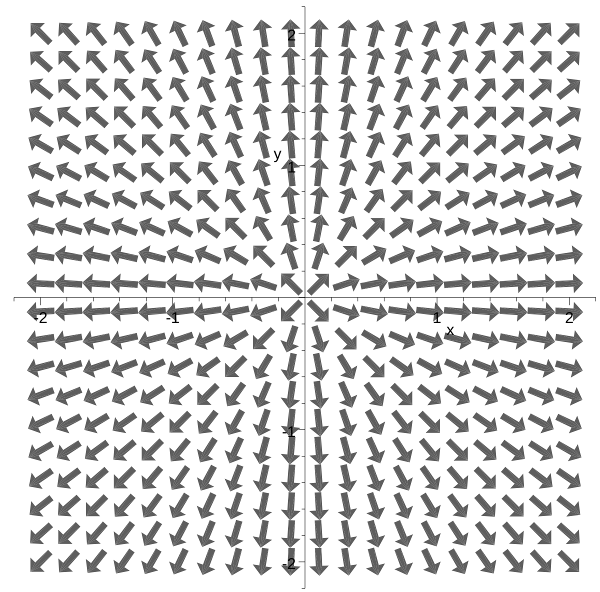
NB No similar definition for $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ for $n > 3$.

Fluid flow interpretation: I

Flow velocity $\mathbf{u} = \alpha(x, y, 0)^{\mathrm{T}}$.





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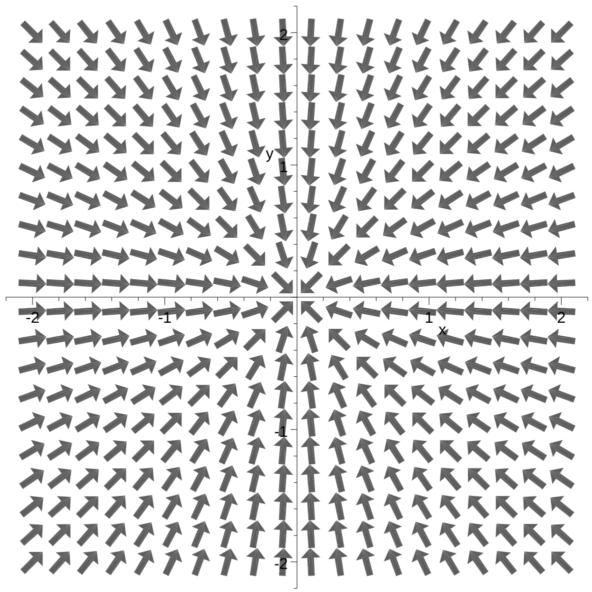
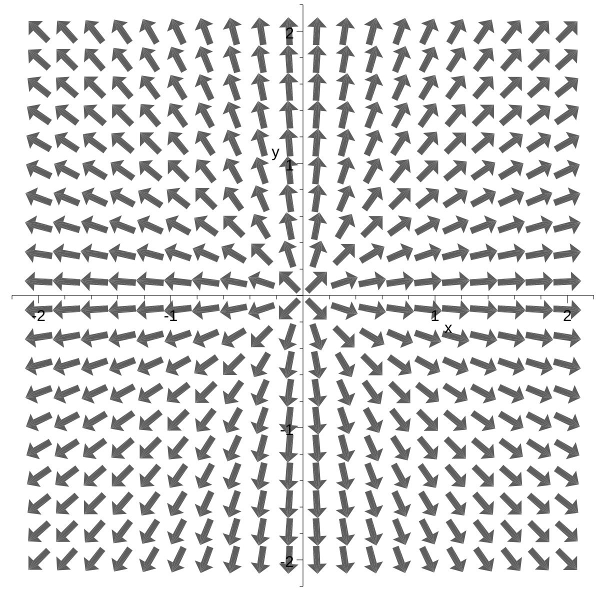
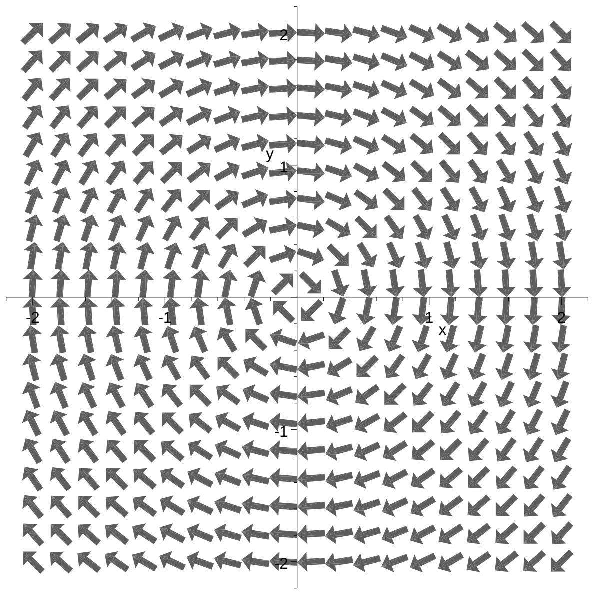


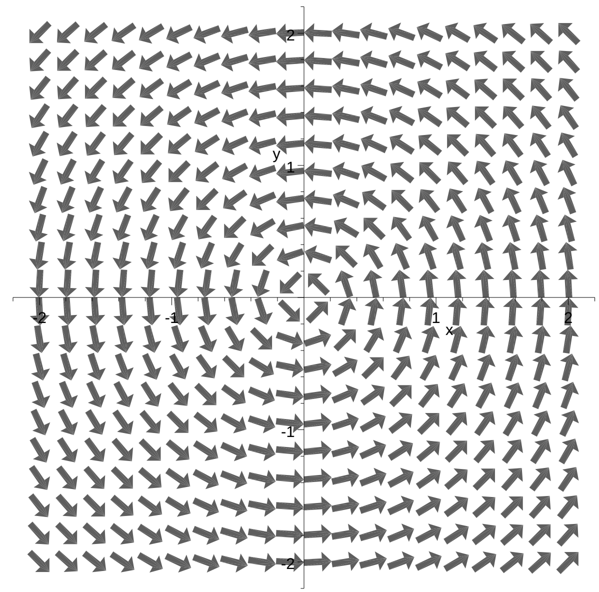


- $\nabla \cdot \mathbf{u} = 2\alpha$, $\nabla \times \mathbf{u} = \mathbf{0}$.
- $\nabla \cdot \mathbf{u}$ measures compression / expansion.
- $\nabla \cdot \mathbf{u} = 0$ indicates incompressible.

Fluid flow interpretation: II

Flow velocity $\mathbf{u} = \alpha(-y, x, 0)^{\mathrm{T}}$.





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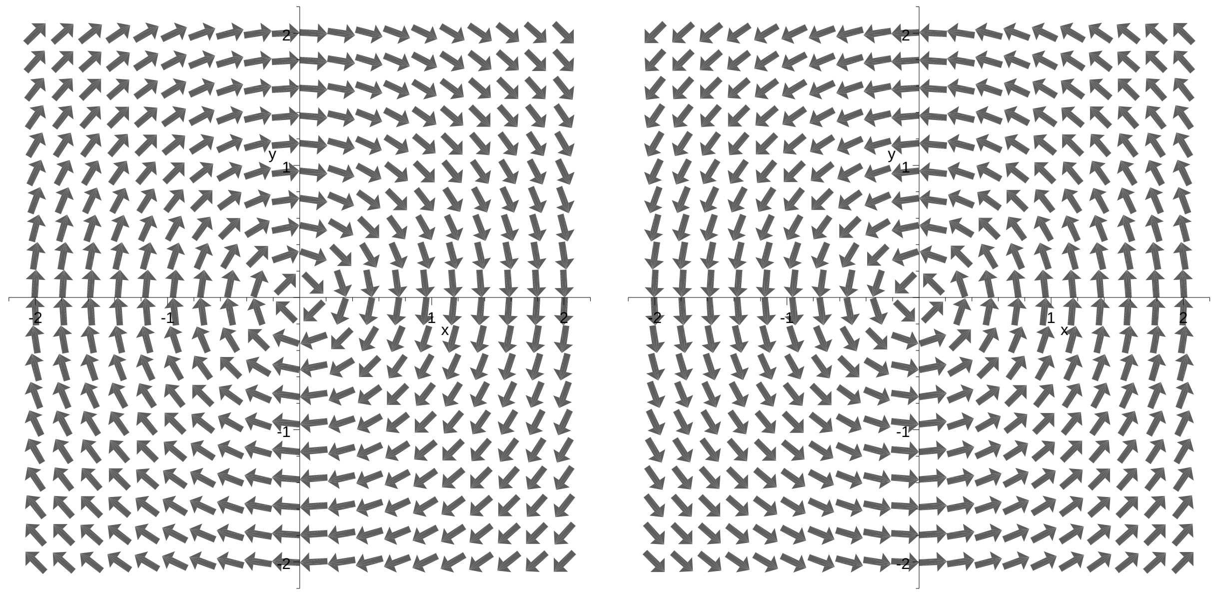

- $\nabla \cdot \mathbf{u} = 0$, $\nabla \times \mathbf{u} = (0, 0, 2\alpha)^{\mathrm{T}}$.
- $\nabla \times \mathbf{u}$ measures rotation.
- $\nabla \times \mathbf{u} = \mathbf{0}$ indicates irrotational.

Basic identities

- *curl grad* $= \mathbf{0}$

$$\nabla \times (\nabla f) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

$$= \begin{pmatrix} f_{zy} - f_{yz} \\ f_{xz} - f_{zx} \\ f_{yx} - f_{xy} \end{pmatrix} \qquad = \mathbf{0}.$$

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- *div curl* $= 0$

$$\nabla \cdot (\nabla \times \mathbf{u}) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{pmatrix}$$

$$= (w_{yx} - v_{zx}) + (u_{zy} - w_{xy}) + (v_{xz} - u_{yz}) \qquad = 0.$$

Homework sheet 2 Question 8

Suppose $f : \mathbf{R}^3 \to \mathbf{R}$, and $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$.

- $\nabla \times (\nabla f) = \mathbf{0}$
- $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
- $\nabla \cdot (\nabla f) = \nabla^2 f$
- $\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F})$
- $(\mathbf{F} \cdot \nabla)\mathbf{F} = \frac{1}{2}\nabla(\mathbf{F} \cdot \mathbf{F}) + (\nabla \times \mathbf{F}) \times \mathbf{F}$

Conservative force fields

- Force field $\mathbf{F}(\mathbf{x})$.
- If $F = \nabla\phi$, then $\mathbf{F}$ is conservative $(-)\phi$ is potential.

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Example:

$\mathbf{F} = (2xy^3,\, 1 + 3x^2y^2, 3z^2)^{\mathrm{T}}$.

$\phi = x^2y^3 + y + z^3 +$ const.

NB $\nabla \times \mathbf{F} = \mathbf{0}$.

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$\phi = x^2y^3 + y + z^3 +$ const.
NB $\nabla \times \mathbf{F} = \mathbf{0}$.

Example:

$\mathbf{F} = (z,\ x,\ y)^{\mathrm{T}}$.
Is there any such ϕ?

$\nabla \times \mathbf{F} = ?$

Coming soon . . .

- more on the chain rule
- ∇ in other coordinate systems
- integral calculus of vector functions