Differential equations project descriptions

(1) **Existence and uniqueness.** (experience with proofs helpful) For this project, you'll prove the existence and uniqueness theorem (Picard's theorem).
   (a) State the Banach contraction theorem, and sketch the proof.
   (b) Apply Banach to prove the E & U theorem. (Don't get bogged down in showing that the Banach hypotheses are met - just sketch the ideas.)
   (c) The proof of E & U given in most differential equations books doesn't use Banach explicitly - it just plows through. This is an instance where introducing some abstraction (contractions in metric spaces) greatly clarifies what's going on. Give some other examples where this occurs. (And, if you feel like it, give some examples where you feel that abstraction obscures what's going on.)

   (A good starting point for this project is Kreyszig, *Introductory Functional Analysis with Applications*.)

(2) **Series solutions to differential equations.** One topic that's often included in an introductory ODE class is the use of power series as solutions.
   (a) Discuss how to use power series to solve ODEs. (Most differential equations texts will have a chapter on this.) Focus on solutions at ordinary points, but briefly discuss regular singular points.
   (b) Discuss the Bessel equation and Bessel functions.
   (c) Is a series really a useful solution for an ODE?
   (d) Why do you think Blanchard, Devaney, and Hall didn't include a chapter on series solutions?

(3) **Difference equations.** (linear algebra helpful) ODEs are often good models for quantities that change continuously. Sometimes, though, change occurs at discrete intervals (every hour, say). In this case, difference equations can be more useful.
   (a) Define difference equations, and describe how they arise. (Give real-world examples. HINT: The Fibonacci numbers are a crowd-pleaser, as is the discrete logistic equation.)
   (b) Discuss methods of solution.
   (c) How is the theory of difference equations similar to that of differential equations? How does it differ?

   (There are whole books written on this topic, so you'll have to be choosy about what information you include.)

(4) **Calculus of variations.** (This might be a little easier if you know what a directional derivative is, but it's not crucial.) Differential calculus is useful for finding the point at which a given function is minimized. The calculus of variations (involving PDEs) is concerned with finding the *function* that minimizes a given quantity. For example, what's the shortest curve joining two given points? (Okay, that's an easy one.) Or, the soap bubble problem: what's the surface with the smallest area that's bounded by a given curve?
   (a) Explain the basic idea of the calculus of variations and give some simple examples.
   (b) One of the most important physical applications of the c. of v. is Hamilton's principle. Discuss.
   (c) Discuss the connection between minimal surfaces and the c. of v. (This is a good place to include some pretty pictures.)

(5) **Chaotic dynamics in the plane.** The behavior of solutions on the real line is so restricted that nothing too interesting can happen, while almost anything you can think of can happen with solutions in three (or more) dimensions. In the plane, though, we get a nice balance; there's plenty of room for strange behavior, but the fact that solutions can't cross is enough of a restriction to impose some useful structure on that behavior.
   (a) Define ω-limit sets.
   (b) State the Poincare-Bendixson theorem, give some examples, and sketch the ideas of the proof.
   (c) Discuss analogs and counterexamples to Poincare-Bendixson on the line and in space.

   (A good place to start is an introductory dynamical systems text, like Alligood, Sauer, and Yorke, *Chaos*.)

(6) **Dirac delta function.** We talked about this briefly, but really just waved our hands at it. For this project, you'll fill in some of the blanks.
   (a) Give a few "definitions" of $\delta(t)$. Which of these are definitions in the mathematical sense of the word?
   (b) Discuss the history and origin of $\delta(t)$.
   (c) In what sense is $\delta(t)$ a function? In what sense is it not? What *is* a function?
   (d) Does $\delta(t)$ have a derivative? Does it have an antiderivative?

   (This is a complicated topic, and it leads to some pretty advanced mathematics. Don't try to do too much.)

(7) **Three-body problem.** Pretend that the solar system consists only of the Earth, moon, and sun, all attracting each other gravitationally the way Newton tells us that they should. That's easy to model as a system of ODEs, but it turns out that it's (really) hard to solve.
   (a) Discuss the much simpler two-body problem, its history, and its solution.
   (b) Discuss the three-body problem. What simplifications have people made (e.g., the restricted planar three-body problem)? How does symmetry simplify the problem?
   (c) Discuss the various ways in which the problem has been "solved."
   (d) Discuss the history of Poincare's mistaken solution, and some of the mathematical consequences of his discovery of his error. (Can you think of other examples of mistakes that turned out to be more interesting than a correct answer would have been?)
   (e) What about the n-body problem?

   (This is a lot, so you'll probably want to treat some of these items very briefly. There's a book on this topic (Barrow-Green, *Poincare and the Three Body Problem*), but I don't know anything about it.)

(8) **Modeling epidemics.** The SIR (susceptible-infected-recovered) model has been widely used to study the spread of disease.
   (a) Discuss the SIR model and some of its variations. What do the various parameters mean physically? What are some of the weaknesses of the mode?
   (b) How do solutions behave? Where are the bifurcations? What does this tell us about how much to spend on immunization vs. treatment vs. prevention?
   (c) Find a case study where the model has been used to examine a real epidemic. How well did the model do?

(9) **How does Maple solve ODEs?** We've talked a little about how computers solve ODEs numerically. But how does Maple solve them symbolically? (I don't know anything about this, so I won't be able to help you much on this project.)
   (a) How does it work?
   (b) Give some examples of ODEs on which Maple works well, and others on which it doesn't work so well. Explain why.

(10) **Comparing different ODE textbooks** (For someone with an interest in teaching.) In the last ten years or so, there's been a big change in the way that differential equations is taught. Our textbook, Blanchard, Devaney, and Hall (BDH), is an example of the new approach.
   (a) The most popular and influential "traditional" ODE book is Boyce and DiPrima. Compare it to BDH. What are the differences? Why are they different? What is each textbook trying to teach the student?
   (b) Discuss the changes in BDH as it's gone from preliminary to first to second to third editions. Why do you think these changes were made?
   (c) Look around on the internet for syllabi from other ODE classes. How many are traditional, and how many are "reformed"?
   (d) Discuss a few specific solution methods that aren't covered in BDH, and give examples of how they work. (You should definitely discuss exact equations, and then perhaps Euler equations or something else.)

   (A good place to start might be the articles at the bottom of http://math.bu.edu/odes/ . I can lend you copies of textbooks.)