1. Consider the system

$$\begin{aligned} x' &= y + y(x^2 + y^2) \\ y' &= -x - x(x^2 + y^2) \end{aligned}$$

The origin $(0, 0)$ is the only equilibrium point. We want to classify the system's behavior near $(0, 0)$.

(a) What does the local linearization tell us about the behavior near $(0, 0)$?

(b) What do the nullclines tell us about the behavior near $(0, 0)$?

(c) Consider the function $f(x, y) = x^2 - y^2$. If $(x(t), y(t))$ is a solution of the system, what is $\frac{d}{dt}(f(x(t), y(t)))$? What does this tell you about the behavior of the system near $(0, 0)$? (Hint: What is the physical meaning of the quantity $x^2 + y^2$?)

a) $x' = y + x^2y + y^3$
$y' = -x - x^3 - xy^2$

$$DF = \begin{pmatrix} 2xy & 1 + x^2 + 3y^2 \\ -1 - 3x^2 - y^2 & -2xy \end{pmatrix}$$

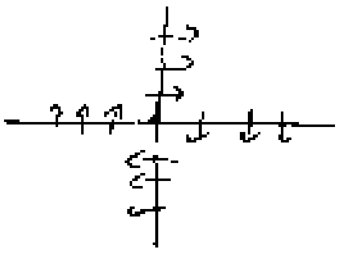
$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Char. poly. is $\lambda^2 + 1$, so eigenvalues are $\lambda = \pm i$

The linearized system has a center at $(0,0)$, so the actual system could have a center, spiral sink, or spiral source

b) x-nullcline: where $0 = x' = y + x^2y + y^3$
$0 = y(1 + x^2 + y^2)$
$1 + x^2 + y^2 > 0$, so x-nullcline is the line $y = 0$

y-nullcline: where $0 = y' = -x - x^3 - xy^2$
$0 = -x(1 + x^2 + y^2)$
$1 + x^2 + y^2 > 0$, so y-nullcline is the line $x = 0$



nullclines tell us nothing new: $(0,0)$ could be a center, spiral sink, or spiral source

c) $\frac{d}{dt}(x^2 + y^2) = 2xx' + 2yy' = 2xy + 2x^3y + 2xy^3 - 2xy - 2x^3y - 2xy^3$
$= 0$

So $x^2 + y^2$ is constant along solns. $x^2 + y^2 = r^2$ is a circle of radius r, so solns. are circles, and $(0,0)$ is a center.

2. The Linearity Principle (aka Principle of Superposition) states that if $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are two solutions of a linear homogeneous system of differential equations, and a and b are constants, then $a\mathbf{Y}_1 + b\mathbf{Y}_2$ is also a solution.
   (a) Give an example showing that the Linearity Principle is not true for linear nonhomogeneous equations.
   (b) Give an example showing that the Linearity Principle is not true for nonlinear equations.

a) Lots of possibilities. Ex: $y' = 1$. Then $y_1 = t$ & $y_2 = t+1$ are solns, but $y_1 + y_2 = 2t+1$ is not.

b) Lots of possibilities. Ex: $y' = y^2$. Then $y_1 = \frac{-1}{t}$ is a soln, but $2y_1 = \frac{-2}{t}$ is not.

3. Consider the second-order ODE $y'' - 3y' - 2y = \overset{10}{\hat{}}\cos t$.

(a) Find the general solution. What is the long-term behavior of solutions?

(b) Find the solution satisfying the initial conditions $y(0) = 3$, $y'(0) = \cancel{3}$ 0.

a) 1st, solve assoc. homog: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$. Gen. sol. is $c_1 e^t + c_2 e^{2t}$

Now, find a particular soln. Guess $y_p = a\cos t + b\sin t$, so

$y_p' = -a\sin t + b\cos t$, &

$y_p'' = -a\cos t - b\sin t$.

plug in: $-a\cos t - b\sin t + 3a\sin t - 3b\cos t + 2a\cos t + 2b\sin t = \cos t$

$(a - 3b)\cos t + (b + 3a)\sin t = 10\cos t$

so $b + 3a = 0$, or $b = -3a$

& $(a - 3b) = 10$

$a + 9a = 10$

$a = 1$, so $b = -3$

$y_p = \cos t - 3\sin t$

Gen soln: $c_1 e^t + c_2 e^{2t} + \cos t - 3\sin t$

Solns go to $\pm\infty$, unless $c_1 = c_2 = 0$, & then they oscillate

b) $y = c_1 e^t + c_2 e^{2t} + \cos t - 3\sin t$, so $y(0) = c_1 + c_2 + 1$

$y' = c_1 e^t + 2c_2 e^{2t} - \sin t - 3\cos t$, so $y'(0) = c_1 + 2c_2 - 3$

solve $3 = c_1 + c_2 + 1$
$0 = c_1 + 2c_2 - 3$ $\Rightarrow$ $c_1 = c_2 = 1$

So $y = e^t + e^{2t} + \cos t - 3\sin t$

EXTRA CREDIT Which is better, a second-order equation or a system of two first-order equations?