1. (25 pts) Today in Evans, they're handing out free samples of a new brand of chewing gum, Bubble Yam® ("the gum that tastes like yams"™). As you chew it, the gum releases sugar, which is dissolved by your saliva. How much sugar is in your mouth after 5 minutes?

The following facts may (or may not) be useful:

- Your salivary glands produce 1 mL of saliva per minute.
- You swallow 1 mL of well mixed saliva per minute.
- t minutes after you start chewing it, the piece of gum is releasing delicious sugar at a rate of e^{-t} mg per minute.
- The color of the gum is neon green.
- Pure saliva contains no sugar.
- The piece of gum weighs 2 g initially.
- There is no sugar in your mouth before you start chewing the gum.

Let $y(t)$ = amt. of sugar in mouth at time t

$$y' = \underbrace{e^{-t}}_{\text{flow in}} - \underbrace{\frac{1\text{mL}}{1\text{mL}} y}_{\text{flow out}} = e^{-t} - y$$

This is linear, so we find an integrating factor.

$$y' + y = e^{-t}$$

$$\underline{\mu y' + \mu y = \mu e^{-t}}, \quad \text{so } \mu' = \mu, \quad \mu = e^{t}$$

$$e^{t} y' + e^{t} y = e^{t} e^{-t}$$

$$(e^{t} y)' = 1$$

$$\int (e^{t} y)' \, dt = \int dt$$

$$e^{t} y = t + c$$

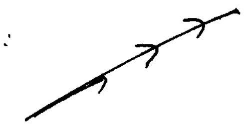
$$y = te^{-t} + ce^{-t}$$

Since $y(0) = 0$, $c = 0$

So $y = te^{-t}$, $\cancel{\&}$ $y(5) = 5e^{-5}$ mg.

2. (10 pts) Describe a class of first-order ODEs for which Euler's method gives completely accurate solutions (no error). HINT: Think about what solution curves would have to look like.

The soln. curves have to be straight lines :



So $y = at + b$, or $y' = a$

3. (15 pts) Show that $y \equiv -1$ is the only solution of the IVP $y' = t(1+y)$, $y(0) = -1$.

$y \equiv -1$ is clearly a solution: $0 = y' = t(1+y) = 0$

Since the partial derivatives of $t(1+y)$ exist & are continuous everywhere, the uniqueness theorem applies, and $y \equiv -1$ is the only soln.

4. (25 pts) Consider the family of differential equations $y' = y + ry^3$, where r is a parameter.

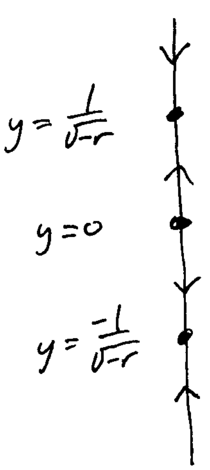
(a) What is the bifurcation value of r?

Find equilibria: $0 = y + ry^3 = y(1+ry^2)$

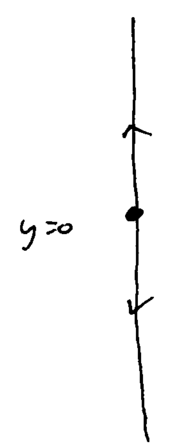
$y=0$ is one equil. If $r<0$, then $y = \pm\frac{1}{\sqrt{-r}}$ gives 2 more equilibria. So $r=0$ is the bifurcation value

(b) Draw phase lines for values of the parameter r slightly smaller than, slightly larger than, and at the bifurcation value. In each case, what are the possible long-term behaviors of solutions?

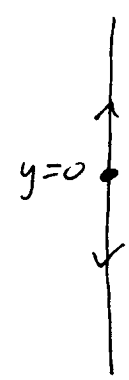
$r<0$



$r=0$



$r>0$



If $y(0)>0$, $y \nearrow \infty$

If $y(0)=0$, $y\equiv 0$.

If $y(0)<0$, $y \searrow -\infty$

If $y(0) > \frac{1}{\sqrt{-r}}$, $y \searrow \frac{1}{\sqrt{-r}}$

If $y(0) = \frac{1}{\sqrt{-r}}, 0,$ or $\frac{-1}{\sqrt{-r}}$, y stays constant.

If $0 < y(0) < \frac{1}{\sqrt{-r}}$, $y \nearrow \frac{1}{\sqrt{-r}}$

If $-\frac{1}{\sqrt{-r}} < y(0) < 0$, $y \searrow \frac{-1}{\sqrt{-r}}$

If $y(0) < \frac{-1}{\sqrt{-r}}$, $y \nearrow \frac{-1}{\sqrt{-r}}$

If $y(0)>0$, $y\nearrow\infty$

If $y(0)=0$, $y\equiv 0$.

If $y(0)<0$, $y \searrow -\infty$

(c) If y is the population of buffaloes in the United States, what do you hope is true about r?

Hope r is fairly small & negative. That way we'll have a fairly large stable population, & it won't grow out of control.

5. (25 pts) Consider the following three models:

- One method of administering a drug is to feed it continuously into the blood stream by a process called intravenous infusion. Here, the drug is continuously pumped into the blood through an IV tube. At the same time, the body's cells absorb the drug out of the blood at a rate proportional to the concentration of the drug in the blood. α, β, and k are positive constants measuring how quickly the drug can be absorbed, how fast the IV fluid is being pumped into the body, and the concentration of the drug in the IV fluid, respectively.
- Many chemical reactions can be viewed as interactions between two molecules that undergo a change and result in a new product. The rate of a reaction, therefore, depends on the number of interactions or collisions, which in turn depends on the concentrations of both types of molecules. Consider the simple reaction $A + B = Y$, in which one molecule of substance A combines with one molecule of substance B to create one molecule of substance Y. α is the initial number of molecules of A, β is the initial number of molecules of B, and k is a positive constant.
- A cake is taken out of a hot oven and placed in a cool room. Newton's law of cooling says that the rate at which the cake's temperature decreases is proportional to the difference between its temperature and the air temperature α. β is the surface area of the cake, and k is a positive constant.

Which equation best matches each model? (One of the equations doesn't match any of the models.) **Explain your reasoning.**

(i) $y(t)$ is the amount of the drug in the bloodstream
(ii) $y(t)$ is the temperature of the cake
(iii) $y(t)$ is the amount of chemical Y

(a) $y' = \beta k(\alpha - y)$
(b) $y' = \alpha k y(\beta - y)$
(c) $y' = k(\alpha - y)(\beta - y)$
(d) $y' = -\alpha y + \beta k$

i - D $\quad y' = \underbrace{-\alpha y}_{\text{absorption into body}} + \underbrace{\beta k}_{\text{drug pumped into body}}$

ii - A $\quad y' = \beta k \underbrace{(\alpha - y)}_{\text{difference b/w air temp \& cake temp.}}$

iii - C $\quad y' = k(\alpha - y)(\beta - y)$

\# molecules of A remaining

\# of molecules of B remaining

\# of collisions is jointly proportional to \# molecules of A & \# molecules of B

EXTRA CREDIT Which college is Agnes Scott's arch-rival, and why?