**Math 118 Midterm #2 Solutions**

**1.** The curve $y^2 = x^3 + 3x^2$ is called the Tschirnhausen cubic (named after Ehrenfried Walther von Tschirnhaus, who hung out with Leibniz and invented European porcelain). Find an equation of the tangent line to his curve at the point $(1, -2)$.

**Implicit differentiation. Set $y = f(x)$. Then $f^2 = x^3+3x^2$, and when we differentiate both sides we get $2ff' = 3x^2+6x$, so $f' = \frac{3x^2+6x}{2f}$, or $dy/dx = \frac{3x^2+6x}{2y}$ $(y \neq 0)$. So at (1,-2), the slope is 9/(-4)=-9/4, and an equation for the tangent line is y-(-2)=-9/4(x-1), or y=-9/4x+1/4.**

**2.** Air is being pumped into a spherical balloon. At time $t$ seconds, the volume of the balloon is $V(t)$ (in cm$^3$) and the radius of the balloon is $r(t)$ (in cm).

(a) Explain in words the meanings of the derivatives $dV/dr$ and $dV/dt$.

(b) If at time $t = 5$ the radius of the balloon is 10 cm and the radius is increasing at a rate of 3 cm/second, how fast is the volume increasing at time $t = 5$? (Reminder: The volume of a sphere is $V = \frac{4}{3}\pi r^3$.)

**(a) $dV/dr$ is in units of cm$^3$/cm. It's the rate of change of volume with respect to the radius; roughly, how much the volume would increase if the radius increased by 1 cm. $dV/dt$ is in units of cm/second. It's how fast the volume is growing with respect to time; roughly, how much the volume would increase after one more second. (b) Chain rule: $dV/dt = dV/dr \cdot dr/dt$. Since $V = \frac{4}{3}\pi r^3$, $dV/dr = 4\pi r^2$. So at $t = 5$, $dV/dt = 4\pi(10^2) \cdot 3 = 1200\pi$ cm$^3$/sec.**

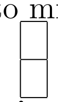
**3.** What is the tangent line approximation to $f(x) = e^{3x}$ near $x = 0$?

**Linear approximation is always $f(x) \approx f(a) + f'(a)(x - a)$. Here, $f(x) = e^{3x}$ and $f'(x) = 3e^{3x}$, so the tangent line approximation is $f(x) \approx 1 + 3x$.**

**4.** Find the absolute maximum and minimum values of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1/2, 4]$.

**Need to check critical points and endpoints. $f'(x) = 3x^2 - 6x$, which exists everywhere and is 0 at 0 and 2. f(-1/2)=1/8, f(0)=1, f(2)=-3, and f(4)=17, so the maximum value is 17 and the minimum value is -3.**

**5.** Farmer Francine wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. Fencing costs \$1 per foot. How can she do this so as to minimize the cost of the fence?

**The field looks like this:**  **If x is the horizontal length and y the vertical length, then we want to minimize the cost 3x+2y. We know that the volume is xy=1,500,000, so y=1,500,000/x, and cost is C(x)=3x+3,000,000/x. The possible values of x are in the interval $(0, \infty)$. We have C'(x)=$3 - 3,000,000/x^2$, which blows up at 0 and is 0 when x is 1000 or -1000, which is not in the interval. C(1000)=6000, and C(x) goes to $\infty$ as x goes to 0 or $\infty$, so she should make her dimensions x=1000, y=1500.**