Math 118 Practice Final Solutions

1. My friend Duncan has decided to enter a donut-eating contest. Let $E(x)$ be the number of minutes that it takes him to eat x pounds of donuts. He has discovered, through months of delicious training and experimentation, that $E(3) = 15$ and $E'(3) = 6$. Roughly how long would it take him to eat 2.5 pounds of donuts? (In other words, use linear approximation to estimate $E(2.5)$.)

$E(x) \approx E(a) + E'(a)(x - a)$. **Here, use** $a = 3$, **so** $E(2.5) \approx E(3) + E'(3)(2.5 - 3) = 15 + 6(-.5) = 12$ **minutes.**

2. Compute the following derivatives:

(a) $\dfrac{d}{dx}(3^x + \ln 3)$

(b) $\dfrac{d}{dx}(\ln(x^2 + 1))$

(c) $\dfrac{d}{dt}(3te^{t^2})$

(d) $\dfrac{d}{dt}\left(\dfrac{2t^5}{1 + 3t}\right)$

(a) $(\ln 3)3^x + 0$ **(remember that** $\ln 3$ **is a constant, so its derivative is 0). (b) chain rule:** $\frac{1}{x^2+1} \cdot (x^2 + 1)' = \frac{2x}{x^2+1}$ **(c) product rule:** $3e^{t^2} + 3t(2te^{t^2}) = (3 + 6t^2)e^{t^2}$ **(d) quotient rule:** $\frac{(1+3t)(10t^4)-2t^5(3)}{(1+3t)^2} = \frac{10t^4+24t^5}{(1+3t)^2}$

3. My uncle, Donald Uriah Tiberius King, has opened a donut store called Donut Czar. He needs your help to figure out how much he should charge for each donut. He pays \$500 a day in fixed costs (rent, wages, bribing health inspectors, etc.). In addition, it costs him \$0.20 to make each donut. After consulting with my brother the economist, Uncle Don has determined that if he charges \$$p$ for each donut, he will sell $x = 5000 - 5000p$ donuts per day.

(a) Find a formula for the daily cost function $C(x)$, which measures the cost (in dollars) of producing x donuts in a day.

(b) Find a formula for the daily revenue R as a function of x (so your answer $R(x)$ should have only x's in it, and no p's).

(c) How many donuts should he make per day to maximize profit?

(d) How much should he charge per donut?

(e) How much money will he make per day?

(a) $C(x) = 500 + 0.2x$ **(b)** $R =$ **(price)(quantity sold)=**px**. Since** $x = 5000 - 5000p$, $p = 1 - \frac{x}{5000}$, **so** $R(x) = (1 - \frac{x}{5000})x = x - \frac{x^2}{5000}$**. (c) Profit** $\Pi(x) = R(x) - C(x) = \frac{4}{5}x - \frac{x^2}{5000} - 500$**. Since he won't charge a negative price, the most he could sell is 5000, so the endpoints are 0 and 5000. Now find the critical points:** $\Pi'(x) = \frac{4}{5} - \frac{x}{2500}$, **so the only critical point is** $x = 2000$**. Check:** $\Pi(0) = -500$, $\Pi(2000) = 300$, **and** $\Pi(5000) = -1500$, **so he should make 2000 donuts per day. (d) To sell 2000 donuts, he should set the price at 1-2000/5000=\$0.60 (e)** $\Pi(2000) =$**\$300**

4. Compute the following integrals:

(a) $\displaystyle\int_{-2}^{1} |x|\,dx$

(b) $\displaystyle\int_{1}^{5} f'(x)\,dx$, if $f(1) = 2$ and $f(5) = 7$.

(a) 2.5 (draw the picture) (b) Fundamental Theorem of Calculus: $\int_1^5 f'(x)\,dx = f(5) - f(1) = 7 - 2 = 5$

5. (20 pts) It is the year 2035, and you're about to be inaugurated as president of Agnes Scott College. Unfortunately, you've overslept. Skipping breakfast, you hop into your nuclear-powered car and speed toward campus. Suddenly, you see a Donut Czar bakery on the side of the road 550 feet ahead of you. Hoping to grab a snack before the ceremony, you slam on the brakes. Will the sudden deceleration send you headfirst through the windshield? Will you be able to stop before you pass the Donut Czar? Perhaps math, along with the following data about your car's speed, can help answer these questions. (Your speed decreases throughout the 8 seconds it takes you to stop, although not necessarily at a uniform rate.)

Time t since brakes applied (sec)	0	2	4	6	8
Speed $v(t)$ (ft/sec)	100	80	50	20	0

(a) Estimate the car's acceleration at time $t = 1$ second after the brakes are applied. What are the units?
(b) Use a left-hand Riemann sum to estimate the distance traveled between the time the brakes are applied to the time the car stops.
(c) Use a right-hand Riemann sum to estimate the distance traveled between the time the brakes are applied to the time the car stops.
(d) Which of the following statements can you justify from the information given? Explain.
  (i) Your car stopped before passing the Donut Czar.
  (ii) The data are inconclusive; your car may or may not have passed the Donut Czar.
  (iii) Your car passed the Donut Czar.

(a) Acceleration is the derivative of velocity: $a(1) = v'(1) \approx \frac{v(2)-v(0)}{2} = \frac{80-100}{2} = -10$, **so you're accelerating at a rate of -10 ft/sec. (b) The distance traveled between the time the brakes are applied to the time the car stops is the integral** $\int_0^8 v(t)\,dt$**, which is approximately** $100\cdot2+80\cdot2+50\cdot2+20\cdot2 =$ **500 ft. (c)** $\int_0^8 v(t)\,dt \approx 80\cdot2+50\cdot2+20\cdot2+0\cdot2 =$ **300 ft. (d) Since** $v(t)$ **is decreasing, we know that the left-hand Riemann sum is an overestimate, so the stopping distance is less than 500 ft., so i) the car stopped before passing Donut Czar.**

6. A delicious hot Donut Czar donut is taken out of the oven and placed on your plate to cool. The temperature H, in degrees Celsius, of the donut t minutes after it was taken out of the oven is given by $H = f(t)$.

(a) Explain in words the meanings of the following equations.
  (i) $f(20) = 100$
  (ii) $f^{-1}(250) = 1$
(b) What are the units of $f'(20)$? What is its practical meaning in terms of the temperature of the delicious donut? Do you expect it to be positive or negative, and why?
(c) What are the units of $(f^{-1})'(250)$? What is its practical meaning in terms of the temperature of the delicious donut?

(a)(i) After 20 minutes, the donut's temperature is 100 degrees. (a)(ii) The donut reaches 250 degrees after 1 minute. (b) Units: degrees/minute. Roughly, it's how much hotter the donut will be after one more minute. It should be negative, since the temperature is decreasing. (c) Units: minutes/degree. Roughly, it's how long it will take for the temperature to increase one more degree.

7. Car A is traveling west at 50 mph and car B is traveling north at 60 mph. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 miles and car B is 0.4 miles from the intersection?

At a given time t, let x be A's distance from the intersection, y B's distance from the intersection, and z the distance between the two cars. We know that $z^2 = x^2 + y^2$, so differentiate both sides to get $2zz' = 2xx' + 2yy'$. Plugging in, we get $2(0.5)z' = 2(0.3)50 + 2(0.4)60$, or $z' = 78$ mph.

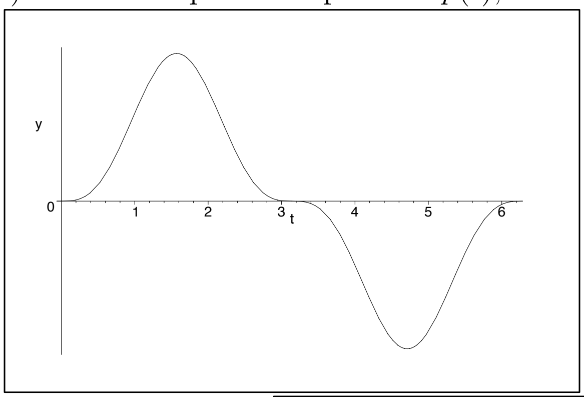
8. Let $f(t)$ be the depth (in feet) of Lake Lanier t days after midnight on January 1. What was the average depth in January?

The average is $\dfrac{\int_0^{31} f(t)\,dt}{31}$.

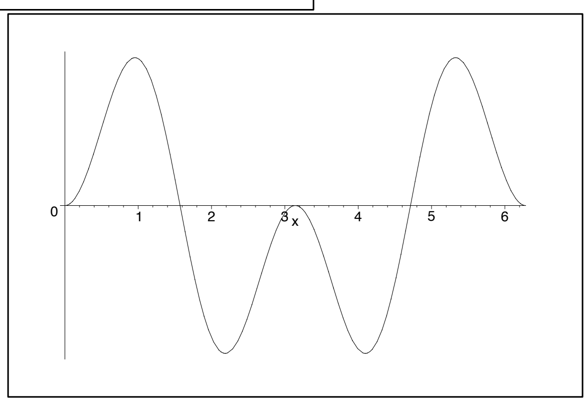
9. The figure below shows the velocity $v(t)$, in cm/sec, of a particle moving along the x-axis. (Right is positive, left is negative.)

(a) Sketch the particle's acceleration $a(t)$.

(b) Sketch the particle's position $p(t)$, assuming that it starts at $x = 0$.



(a) acceleration:



(b) position: